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2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
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The Tower

1. And the whole science was of one form and of one notation.

2. And it came to pass as they who studied the science journeyed through it, that they found a common ground of understanding, and enlarged the wisdom of mankind.

3. And they said to one another, Go to, let us make a book and engrave it fairly. And they had words for ideas, and symbols they had for convenience.

4. And they said, Go to, let us build us a library of knowledge whose top may reach unto Heaven; and thus let us make us a name befitting ourselves, lest we be underestimated throughout the earth.

5. And the Lord came down to see the thing that the students of mathematics builded.

6. And the Lord said; Behold this people is one and they have all one symbolism; and this they begin to do: and now nothing will be restrained from them, which they have imagined to do.

7. Go to, let us go down, and there confound their terminology, that they may not understand one another.

8. And the Lord did.

9. Then began each man to use terms of his own fancy, and to devise his own notations.

10. And each looked into the faces of the others aghast, and uncomprehending what they said.

11. And they gathered themselves into small schools and departed away from one another.

12. So the Lord scattered them abroad from thence upon the face of all the earth, and they left off to build.

N. E. Rutt.

(With apologies to Moses.)

The Representation of Integers in Forms[1]

By Ralph Hull
The University of British Columbia

1. *Introduction.* The number of all representations of $2^k m$, m a positive odd integer, $k \geqq 0$, by the quadratic form x^2+y^2 is $4E(m)$, where $E(m)$ is the excess of the number of positive divisors of m of the form $4t+1$, that is, $\equiv 1 \pmod 4$, over the number of positive divisors $\equiv 3 \pmod 4$ of m. For example, when $k=1$, $m=65$, we have $E=4$, since the positive odd divisors of 65 are 1, 5, 13, 65, all $\equiv 1 \pmod 4$, and it is easily verified by trial that there are exactly $4 \times 4 = 16$ distinct pairs of integers (x, y) such that $x^2+y^2=130$, counting such pairs as $(3, 11)$, $(-3, -11)$ or $(7, 9)$, $(9, 7)$, etc., as distinct. Again, when $k=0$, $m=35$, we have $E=0$. There do not exist integers x and y such that $x^2+y^2=35$; in other words, 35 is not represented by the form x^2+y^2.

A positive integer is representable as a sum of three squares if and only if it is not of the form $4^k(8t+7)$, $k \geqq 0$, $t \geqq 0$. The number of representations as a sum of three squares is also expressible in terms of the divisors of the number represented, but none of the various expressions is as simple as that above, first obtained by Jacobi about 1829, for the sum of two squares. Lagrange was the first to prove that *every* positive integer can be represented as a sum of four squares. The number of representations of $m>0$ in the form $x^2+y^2+z^2+w^2$ is 8 times the sum of the positive odd divisors of m when m is odd and 24 times this sum when m is even.

The theorems quoted have been chosen for their own intrinsic beauty and completeness, as well as for purposes of illustration of the more general theory of the representation of integers in forms. The extent of this theory may be gauged from the fact that the entire third volume of Professor L. E. Dickson's *History of the Theory of Numbers* is devoted to it, while much of the two earlier volumes is concerned with related problems, treated in a less general or systematic way. Contributions to the theory have been made by many of the most prominent mathematicians of the last two or three centuries, and the extent of the theory is equalled by the variety of its results and

[1] This is the third article in a series of expository articles solicited by the editors.

methods. Two topics, ***Quadratic Forms*** and ***Waring's Problem***, will be described at some greater length in later sections: the first, because it is one of the best examples in the theory of numbers of the power of general methods; the second, because it is of especially timely interest in that it is only within the last three or four years that efforts to attain the "ideal" have been successful. Both topics illustrate how two distinct methods, the algebraic and the analytic, supplement each other to yield complete results. Important new contributions to the theory of quadratic forms, as well as to the solution of Waring's problem, have been made quite recently. It is not the aim of this exposition to provide a specialist with a compendium of results for reference, but a few suggestions regarding some of the more important and active branches of the general theory of representation will be given before these two topics are enlarged upon.

We shall understand by a *form*, a homogeneous polynomial in a certain number of variables, whose coefficients are rational integers unless otherwise specified. The word *representation*, without a modifier, means an integral representation as illustrated above. Occasional reference will be made to rational representation, real representation, etc. As is to be expected, the problems which arise, and the difficulties involved in their solution, vary with the degree and with the number of variables. Emphasis may be placed variously upon determining conditions on an integer in order that a given form shall represent it, upon finding expressions for the number of representations, upon methods of finding representations, upon proving impossibility of representation, upon proving universality, etc. It is usually a related, but by no means equivalent problem, to determine rational representations, or p-adic representations. This will be illustrated later.

2. *Related Topics*. In connection with linear forms, space permits only brief mention of the *theory of partitions*, that is, the problem of determining the number of ways in which a given positive integer can be expressed as a sum of positive integers. Asymptotic formulas for the number of partitions of a given integer have been obtained by Hardy and Ramanujan[2] to whom we refer the reader for further details and references. This problem is of importance in some connections in physics, for example, as shown by van Lier and Uhlenbeck,[3] in the calculation of the density of the energy levels of nuclei.

The theory of binary quadratic forms is very closely related to the theory of the ideals of quadratic algebraic number fields, as, for

[2] Hardy and Ramanujan, Proceedings of the London Mathematical Society (2), 17 (1918), pp. 75-115.

[3] Van Lier and Uhlenbeck, Physica, 4 (1937), pp. 531-542.

example, the *norm form* x^2+y^2 is related to the field $R(i)$, where R is the rational field and $i^2=-1$. Similarly, certain quaternary quadratic forms are related to generalized rational quarternion algebras. Through his studies of such forms, Brandt was led to the fundamental theorems concerning the ideals of normal simple algebras.[4)] Conversely, through the study of the arithmetic of an algebraic number field, or a normal simple algebra, one is led to theorems concerning representation of integers by the corresponding norm forms. For example, the norm form of the algebra of ordinary rational quaternions is a sum of four squares, and the result quoted in §1 concerning that form can be derived from the arithmetic of the algebra.[5)]

There is as yet no general theory of forms of degree >2 as extensive or as complete as that for quadratic forms, although many interesting results have been obtained. One of these, for example, is the theorem of Thue: *if* $f(z)=a_nz^n+\cdots+a_0$ *is an irreducible polynomial relative to* R, *of degree* $n\geqq 3$, *with integral coefficients, then any integer whatever has at most a finite number of representations in the form* $a_nx^n+a_{n-1}x^{n-1}y+\cdots+a_0y^n$. This result should be contrasted with the case of a *Pell form* such as x^2-5y^2, which represents 4 in infinitely many ways. Nagell[6)] has outlined the work of Thue, Siegel, and others in this and similar connections, and also gives an extensive bibliography.

Various problems concerning representation and rational representation by binary forms of higher degree are at present being attacked from another point of view by Hasse[7)] and others. The study of algebraic function fields of one variable, with finite cofficient fields, is directed largely toward this end. The theory of the divisors of algebraic function fields, the Riemann-Roch Theorem, etc., originally developed with the ordinary complex field as coefficient field, have been abstracted and made available for this purpose. As remarked by Nagell[8)], it is the genus of the associated curves, rather than the degree, which determines the nature of the results and difficulties.

We conclude this section by pointing out a generalization of which nearly all the problems of representation are susceptible. One may evidently replace the rational integers by the integers of an algebraic number field and raise the same questions concerning representation,

[4] For the literature on linear associative algebras, see Deuring, *Algebren*, Ergebnisse der Mathematik, IV, 1 (1935), Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, XXIV (1939).

[5] Cf. Dickson, *Algebren und ihre Zahlentheorie*, pp. 154-183.

[6] Nagell, *L'analyse indéterminée de degré supérieur*, Mémorials des Sciences Mathématiques, Fasc. XXXIX (1929).

[7] See Hasse's descriptive article, Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Math. Phys. Kl., III, 18 (1937), pp. 51-55.

[8] Cf. Nagell, op. cit. 6), p. 3.

relative to such a number system. In problems of proving impossibility of representation, this generalization may yield results for ordinary integers as corollaries, as in the proofs in certain cases of the impossibility in integers of the Fermat equation $x^n+y^n=z^n$, $n>2$.[9] The theory of quadratic forms in several variables, as developed by Hasse and Siegel,[10] is to a large extent valid for such number systems as well as for the rational integers.

3. *Quadratic Forms.* Fermat stated many theorems concerning the representation of integers in special binary quadratic forms, as, for example, that x^2+3y^2 represents all primes of the form $6t+1$ but no prime of the form $6t-1$. As a rule he did not give proofs, although he claimed to have them, and many of these were supplied by Euler. Lagrange was able to go much further by means of his more general theory, and Legendre, further still, aided by his quadratic reciprocity law. The general theory of quadratic forms as it exists today, however, originated largely in the work of Gauss.

The reader will find expositions of the elementary theory, as simplified and extended by many later writers, in Dickson's *Introduction to the Theory of Numbers* and *Studies in the Theory of Numbers,* and in Landau's *Vorlesungen über Zahlentheorie,* Bd. I. The first of these may also be consulted for the fundamental properties of congruences, quadratic residues, etc., familiarity with which is assumed in this and the next section. The second deals with quadratic forms in several variables and contains, besides simplified expositions of earlier work such as that of Meyer on indefinite ternaries, many results of Dickson's own research, and that of his students, in this field. The third provides an alternative introduction to the theory of numbers and emphasizes the elementary analytic theory at an early stage. In particular, Landau's text may be consulted for the analytic determination of the class number of binary quadratic forms, and for a proof of Dirichlet's theorem on primes in arithmetical progressions, which is almost indispensable in the advanced theory.[11] Proof by infinite descent, which Fermat claimed that he possessed for many of his theorems, is illustrated in one proof[12] of the impossibility in integers >0, of the equation $x^4+y^4=z^2$. Students in the theory of numbers

[9] Cf. Landau, *Vorlesungen über Zahlentheorie,* Bd. III, pp. 207-210, for the case $n=4$.

[10] See §4.

[11] Since the preparation of this exposition Dickson's new book: *Modern Elementary Theory of Numbers* (University of Chicago Press, 1939) has appeared. All of the topics mentioned above in connection with his two earlier texts are treated in the new text, in many instances more simply than hitherto. In addition, there is a chapter on Waring's problem, and, in an appendix, a proof of the Dirichlet theorem on primes in arithmatical progressions.

[12] Cf. Dickson, *Introduction to the Theory of Numbers,* pp. 42-43.

are fortunate in having available for reference the three volumes of Dickson's *History of the Theory of Numbers*. The reader is urged to read the preface to the third volume, and at least to glance over the first few chapters of that volume.

Jacobi's theorem[13] on the form x^2+y^2 is an elementary one although his proof depends upon formulas relating to elliptic functions, in particular, upon comparision of the expressions

$$\frac{2K}{\pi}=1+4\sum_{m'=1}^{\infty}A(m')q^{m'}, \qquad \left(\frac{2K}{\pi}\right)^{\frac{1}{2}}=\sum_{n=-\infty}^{\infty}q^{n^2},$$

where $$K=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \quad K'=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k'^2\sin^2\varphi}},$$

with $|k|<1$, $k'=(1-k^2)^{\frac{1}{2}}$, $q=e^{-\pi K'/K}$, and where $A(m')=E(m)$, $m'=2^h m$, $h\geqq 0$, m odd. We shall indicate how Jacobi's theorem follows from a more general result of Dirichlet.

To this end, consider the binary quadratic form

(1) $$f=ax^2+bxy+cy^2,$$

where a, b, and c are integers. The integer $d=b^2-4ac$ is called the *discriminant* of f, and the g. c. d. of a, b, and c is called the *divisor* of f. When its divisor is 1, f is said to be *primitive*, and otherwise, *imprimitive*. An equation

(2) $$a\alpha^2+b\alpha\gamma+c\gamma^2=m,$$

where α and γ are integers, is said to be a *proper* representation of m by f when the g. c. d. of α and γ is 1, and otherwise, *improper*. Assume that (2) is a proper representation of m by f. Then there exist integers β and δ such that $\alpha\delta-\beta\gamma=1$. The transformation

(3) $$x=\alpha X+\beta Y$$
$$y=\gamma X+\delta Y,$$

replaces f by the form $F=mX^2+nXY+lY^2$, which is said to be *equivalent* to f, since it is obtained from f by an integral transformation of determinant 1. It is easily seen that two equivalent forms have the same discriminant so that the coefficients of F satisfy the relation $n^2-4lm=d$. This indicates a correspondence between proper representations of m by f and solutions of the congruence

(4) $$n^2\equiv d \pmod{4m}.$$

[13] For one direct elementary proof of this theorem, based on the properties of Farey fractions, see Landau, *Vorlesungen über Zahlentheorie*, Bd. I, pp. 101-105.

This correspondence is unique if we restrict n further to satisfy

$$(5) \qquad 0 \leqq n < 2m.$$

Conversely, to find a proper representation of m by a given form f of discriminant d, we seek first a solution of (4) and (5). If no solution exists, there is no representation of m by f. If n is a solution of (4) and (5), we obtain also an integer $l = (n^2 - d)/4m$, and a form $F = mX^2 + nXY + lY^2$, of discriminant d. We have then to determine whether or not F is equivalent to f. If not, there is no representation of m by f corresponding to the particular root n of (4) and (5) with which we started. If F is equivalent to f, let the transformation:

$$(6) \qquad \begin{aligned} X &= \delta x - \beta y \\ Y &= -\gamma x + \alpha y, \end{aligned}$$

where $\alpha\delta - \beta\gamma = 1$, carry F into f. Then (2) is a proper representation of m by f, corresponding to the root n of (4) and (5).

It is here that the connection between the problem of representation, and that of equivalence in the arithmetical sense defined above, becomes apparent. The latter problem may be solved in any particular case by means of the theory of *reduced forms* or *reduction*. We call the totality of forms equivalent to a given form a *class* of forms. All forms of one and the same class have the same discriminant and represent exactly the same integers. In general, the forms of a given discriminant fall into several classes, the number of which is finite, and is called the *class-number* of the discriminant. The class-number has been the subject of a great deal of research. For example, recursion formulas for its determination, known as *class-number relations*, have been obtained.[14] There are various methods, many of them geometrical, of selecting from each class one representative form, or perhaps several forms, satisfying special conditions, and called *reduced*. By means of some process of reduction by which we can readily determine the reduced form or forms to which two given forms are equivalent, we can decide whether or not the two forms are equivalent.

A binary quadratic form is said to be *definite* if its discriminant is negative. Then the form represents only non-negative or only non-positive integers. Let f be definite and assume also that $a > 0$, so that f is ***positive definite*** and represents only non-negative integers. Such a form is said to be reduced if $-a < b \leqq a$, $c \geqq a$, $b \geqq 0$ if $c = a$. There is one and only one reduced form in each class of such forms. For ex-

[14] One of the most fruitful methods of finding class-number relations is that of "paraphrase". Cf. Bell, *Algebraic Arithmetic*, American Mathematical Society Colloquium Publications, Vol. VII.

ample, in the case $d=-4$, there is a single class and the reduced representative of the class is the form x^2+y^2.

In addition to the proper representations of m by f, there may be improper representations of m by f. These correspond in an obvious way to proper representations by f of factors of m of the type m/m_1^2, $m_1>1$, when such exist, and they are related to congruences like (4) and (5) with suitable moduli. In the case of positive definite forms, to each root of such a congruence there corresponds a finite number w of distinct representations by some form of the discriminant. The value of w is 6 for forms in the class of x^2+xy+y^2; $w=4$ for forms in the class of x^2+y^2; and $w=2$ for all other primitive, positive, definite forms. The form x^2+y^2, for example, has $w=4$ *automorphs*, that is integral transformations of determinant 1 which carry the form into itself. These are: $x=X$, $y=Y$; $x=-Y$, $y=X$; $x=Y$, $y=-X$; $x=-X$, $y=-Y$. Elementary theory gives us the total number of solutions of the congruences (4) and (5) involved, and this number, multiplied by w, is essentially the formula of Dirichlet[15)] for the total number of representations of m by the various forms of a representative system of the positive, primitive, integral forms of discriminant d, one form, for example, the reduced form, being chosen from each class. Since there is only one class for $d=-4$, all the representations necessarily hold for the one form x^2+y^2 and Dirichlet's formula yields Jacobi's in this case. It yields similar formulas in other cases also, but, in general, gives only the total number of representations by a set of forms.

In the case of certain discriminants for which there is more than one class, arithmetical invariants, called characters, enable one to distinguish between the integers represented by one class and those represented by another class. The characters are defined by means of Legendre quadratic character symbols in the case of binary forms, and lead to the concept of a *genus*, which will be defined otherwise in the next section. The discriminants for which the distinction mentioned can be made are those for which there is a single class in each genus.

There is a corresponding theory for *indefinite* binary forms, that is, those with positive discriminants. One difference is that such forms have an infinitude of automorphs, corresponding to representations of 4 by Pell forms mentioned in §2. The theory of reduction also diverges widely from that for definite forms.

It is convenient to employ matrix notation in dealing with quadratic forms in several variables. The form

$$a_{11}x_1^2+2a_{12}x_1x_2+\cdots+a_{rr}x_r^2,$$

[15] Cf. Dickson, op. cit. 12), p. 78.

which has even coefficients with the products $x_i x_j$, $i \neq j$, may be written as the matrix product xAx', where x is the row-vector $(x_1, \cdots, x_r)$; A is the symmetric matrix $A = (a_{ij})$, with integral elements $a_{ij} = a_{ji}$, called the matrix of the form; and the accent denotes the transpose, as usual.

Two forms, xAx' and yBy', are said to be *arithmetically equivalent* if there exists an integral matrix P, of determinant 1, such that xAx' becomes yBy' through the transformation $x = yP$. Then $B = PAP'$, so that A and B are *congruent* matrices. In particular, congruent matrices are equivalent with respect to *elementary transformations*[16)] of matrices in the integral domain of rational integers. Hence, if A and B are arithmetically equivalent they have the same invariant factors relative to such elementary transformations. Let $i_1, \cdots, i_r$ be the invariant factors of A, arranged in the usual order so that i_{k-1} divides i_k, $k = k = 2, \cdots, r$. Then $o_0, o_1, \cdots, o_{r-1}$, where $o_0 = i_1$, $o_j = i_{j+1}/i_j$, $j = 1, \cdots, r-1$, are the order invariants of Poincaré and Minkowski.[17)] There are r additional order invariants[17)] $\sigma_1, \cdots, \sigma_r$, where $\sigma_j = 1$ or 2, according as there is or is not a principal submatrix of A of degree j with odd determinant. These $2r$ order invariants are arithmetical invariants in place of which we employ only the divisor and the discriminant in the binary case. Another invariant relative to arithmetical equivalence, besides the number of variables, is the *index* of the form, that is the number of positive terms in the *real* canonical form as a sum of squares with coefficients equal to 1 or -1. In the binary case, we have positive definite, indefinite, and negative definite forms according as the index is 2, 1, or 0, respectively. Quadratic forms in the same number of variables and with the same index are said to belong to the same *order* if they have the same order invariants. The orders are sub-divided into genera and the genera into classes.

The order invariants may be defined in terms of the g. c. d.'s of the determinants of sub-matrices of the matrix of a form, and they are related to the companion matrices of the matrix. In the ternary case, the only non-trivial companion matrix is the adjoint matrix, and the form associated with the matrix obtained from the adjoint matrix by removing a certain common factor of its elements is called the *reciprocal* form. It is found that the representations of an integer by a ternary form are closely connected with the representations of certain binary forms by the reciprocal ternary form.[18)] In the matrix notation

[16] Cf. Albert, *Modern Higher Algebra*, pp. 59-61, 70-74.

[17] The definitions are essentially Poincaré's, Comptes Rendus Paris, 94 (1882), pp. 67-69, 124-127; the notations are Minkowski's, Mèmoires divers savants Institut de France (2), 29 (1884), No. 2; Gesammelte Abhandlungen, I, pp. 157-202. For other references, see Dickson, *History of the Theory of Numbers*, Vol. III, Chapter XI.

[18] Cf. Dickson, *Studies in the Theory of Numbers*, p. 33.

this idea of the representation of one form by another is an obvious extension of that of the representation of an integer. Thus, a form xAx' is said to represent a second form yBy', where the degree of B is equal to or smaller than that of A, if there exists an integral matrix Q, whose column number is the degree of A and whose row number is the degree of B such that $QAQ'=B$. This specializes to the representation of an integer if $B=(m)$ is a matrix of degree 1. The work of Hasse and Siegel, to be described briefly in the next section, deals with this extension of the idea of representation.

4. *Analytical Theory of Quadratic Forms.* We have already referred to the analytic determination of the class-number of binary quadratic forms and have indicated a connection between such forms and the elliptic functions of Jacobi. It is the purpose of this section to derive the theorems on sums of two, three, and four squares, which were stated in §1, from the recent, general, analytic theorems of Siegel[19] on representation. Class-number formulas, and many other previously known analytic results, are included in Siegel's theorems, and he develops the connections between quadratic forms and modular functions which correspond to that with elliptic functions in the special case. We shall first indicate certain results due to Hasse[20] on rational representation. Both Hasse and Siegel deal with the representation of one form by another, including the representation of integers as a special case. Hasse studies also "rational" representation in an algebraic number field and Siegel, the corresponding "integral" representation.

We have defined the representation of one form by another at the end of §3. When the matrix Q is required only to have rational elements, not necessarily integral, or real elements, not necessarily rational, we speak of *rational* and *real* representation, respectively. When the symmetric matrices A and B have integral elements, we say that the form xAx' *represents* the form yBy' *modulo* q if there exists an integral matrix Q such that $QAQ'\equiv B \pmod{q}$, where the matrix congruence means elementwise congruence. With these definitions we may state the result of Hasse, from which Siegel starts, essentially as follows:

An integral form xAx' represents rationally the integral form yBy' if and only if there exists a real representation of yBy' by xAx' and a representation modulo q of yBy' by xAx' for every modulus q.

[19] Siegel, Annals of Mathematics (2), 36 (1935), pp. 527-606. This paper deals chiefly with definite forms. Siegel treats the indefinite case in two later papers, ibid., Vol. 37 and Vol. 38.

[20] Hasse, Journal für Mathematik, 152 (1923), pp. 129-148, 205-224; 153 (1924), pp. 12-43, 113-130, 158-162.

This criterion for rational representation is a generalization of a result of Legendre for the binary case and it can also be expressed in terms of p-adic numbers. When A and B have the same degree the theorem yields a criterion for rational equivalence as a special case, that is, for the existence of a non-singular matrix P with rational elements such that $PAP' = B$. Rational equivalence, however, does not in general imply arithmetical equivalence. Hasse calls the totality of forms which are rationally equivalent to a given form a *complex* of forms. Siegel, following Poincaré and Minkowski, defines a *genus* of quadratic forms as the totality of quadratic forms which are equivalent to each other by a real transformation and also equivalent modulo q for every modulus q. We call two forms xAx' and yBy' *equivalent modulo* q if there exists an integral matrix Q with determinant $\equiv 1$ (mod q) such that $QAQ' \equiv B$ (mod q). Minkowski proved that two integral quadratic forms are in the same genus if and only if there is a rational matrix of determinant 1, and such that the l. c. m. of the denominators of its elements is prime to twice the product of the determinants of the two forms, which carries one form into the other. From these definitions and results it follows that two forms which are rationally equivalent are not necessarily arithmetically equivalent unless there is a single class in the complex to which the forms belong. In particular, the forms would then belong to the same genus and the genus would consist of a single class.

Siegel's fundamental theorem on representation by positive, definite, quadratic forms concerns the number of representations of an integer, or of another quadratic form, by representatives of the classes belonging to a genus. We obtain from it a formula for the number of representations by a single form when there is a single class in the genus. This is known to be the case for the genera of each of the three forms:

$$(7) \qquad S_2 = x^2 + y^2, \quad S_3 = x^2 + y^2 + z^2, \quad S_4 = x^2 + y^2 + z^2 + w^2,$$

in which we are particularly interested. We shall state only the special case of the theorem pertaining to the number of representations of an integer by the forms of a genus.

Consider a genus of positive, definite, integral quadratic forms in r variables. Let h be the number of classes in the genus, and let $f_1, \cdots, f_h$ be any representatives of the classes in the genus, one form from each class; let $A(f_i, t)$ and $G(f_i)$, $i = 1, \cdots, r$, denote the number of representations of the integer $t > 0$ by f_i and the number (finite) of automorphs of f_i, respectively. Then Siegel's theorem states that:

$$\text{(8)} \qquad \frac{\displaystyle\sum_{i=1}^{h} \frac{A(f_i,t)}{G(f_i)}}{\displaystyle\sum_{i=1}^{h} \frac{1}{G(f_i)}} = \nu \cdot \lambda(f,t) \prod_{(p)} d_p(f,t),$$

where $\nu = 1/2$ when $r=2$ and $\nu = 1$ when $r = 3$ or 4;

$$\text{(9)} \qquad \lambda(f,t) = \frac{\pi^{r/2}|f|^{-\frac{1}{2}} t^{r/2-1}}{\Gamma(r/2)}$$

is a quantity defined in terms of certain volumes and called by Siegel the *density* of representation of t by any form f of the genus, and $|f|$ is the determinant of f;[21]

$$\text{(10)} \qquad d_p(f,t) = \lim_{n\to\infty} \frac{M(p^n,f,t)}{p^{n(r-1)}}$$

is the *p-adic density* of representation of t by any form f of the genus, and $M(p^n,f,t)$ is the number of solutions of the congruence

$$\text{(11)} \qquad f \equiv t \pmod{p^n};$$

and, finally, the product is over all rational primes p in their natural order and converges, at least conditionally, so that it vanishes if and only if a factor vanishes.

The left member of (8) reduces to $A(f_1,t)$ when $h=1$, as for the genera of S_2, S_3, and S_4. We insert the value of λ from (9) with $|f|=1$, and the appropriate value of ν, and obtain the formulas:

$$\text{(12)} \qquad A(S_2,t) = \frac{\pi}{2} \prod_p d_p(S_2,t),$$

$$\text{(13)} \qquad A(S_3,t) = 2\pi t^{1/2} \prod_p d_p(S_3,t),$$

and

$$\text{(14)} \qquad A(S_4,t) = \pi^2 t \prod_p d_p(S_4,t).$$

respectively.

In order to evaluate the numbers d_p it is necessary to determine the numbers of solutions of congruences like (11). This is an elementary problem and it has been considered by a number of writers. The ratio whose limit is required in (10) is constant for n sufficiently large, as is illustrated below in the determination of $d_2(S_3,t_1)$, t_1 odd. It is con-

[21] Cf. Siegel, Annals of Mathematics (2), 36 (1935), p. 568. We have written $\lambda(f,t)$ for Siegel's $A_\infty(\mathfrak{S}, \mathfrak{T})$.

venient at this point to consider the forms S_2, S_4, and S_3, in that order, in the separate cases (i), (ii) and (iii) below.

Case (*i*). $S_2=x^2+y^2$. If p is an odd prime, it is found that[22)]

$$d_p(S_2,t)=(1-\epsilon_p/p)(1+\epsilon_p\cdots+\epsilon_p{}^{\alpha_p}), \tag{15}$$

where $\epsilon_p=(-1|p)=(-1)^{(p-1)/2}$ is the Legendre symbol, and where p^{α_p} divides t, p^{α_p+1} does not divide t, $\alpha_p\geqq 0$. On substituting the values (15) in (12), we obtain

$$A(S_2,t)=\frac{\pi}{2}d_2(S_2,t)PQ(t), \tag{16}$$

where
$$P=\Pi'(1-\epsilon_p/p),$$

the accent denoting the product over all *odd* primes, and where

$$Q(t)=\Pi''(1+\epsilon_p+\cdots+\epsilon_p{}^{\alpha_p}), \tag{17}$$

the double accent denoting the product over all odd primes which divide t. The quantity P is independent of t and it can be shown to be the reciprocal of the series

$$1-1/3+1/5-\cdots=\pi/4. \tag{18}$$

The value $\pi/4$ for the series (18) is known from the elementary calculus: arctan $(1)=\pi/4$. Formula (12) provides an alternative arithmetical proof of (18). For example, take $t=5$, evaluate $A(S_2,5)$, $d_2(S_2,5)$ and $Q(5)$, and substitute in (12). The quantity $Q(t)$ has the value $E(t_1)$, where E is the function defined in §1, $t=2^\alpha t_1$, $\alpha\geqq 0$, t_1 odd. The reader can verify this by multiplying out, replacing products of Legendre symbols by Jacobi symbols,[23)] and noting that $t_1=\Pi''p^{\alpha_p}$. It may be noted in passing that $Q(t)=Q(t_1)=0$ in case α_p is odd for a prime $p\equiv 3 \pmod 4$ which divides t, since $p\equiv 3 \pmod 4$ implies $\epsilon_p=-1$ and $1+\epsilon_p+\cdots+\epsilon_p{}^{\alpha_p}=1$ or 0 according as α_p is even or odd.

The direct evaluation of $d_2(S_2,2^\alpha t_1)$, t_1 odd, is tedious for large α and we evade this in case t is even, that is, $\alpha>0$, as follows.[24)] Suppose $x^2+y^2=2m$, x, y and m integers. Then x and y are both even or both odd so that $x+y=2X$, $x-y=2Y$, where X and Y are integers such that $X^2+Y^2=m$. Conversely, $x=X+Y$ and $y=X-Y$ are uniquely determined by X and Y and $X^2+Y^2=m$ implies $x^2+y^2=2m$. In this way we see that $A(S_2,2m)=A(S_2,m)$ and, by induction, that

[22] Cf. Siegel, l. c. 21), *Hilfssatz* 16.
[23] Cf. Dickson, op. cit. 12), p. 36.
[24] Cf. Dickson, op. cit. 12), p. 80.

$A(S_2,2^\alpha t_1)=A(S_2,t_1)$. Hence we need only evaluate $d_2(S_2,t_1)$, t_1 odd. We shall show that

$$(19) \qquad d_2(S_2,t_1) = \lim_{n\to\infty} \frac{M(2^n,S_2,t_1)}{2^n} = \frac{M(2^3,S_2,t_1)}{2^3},$$

where $M(2^n,S_2,t_1)$ is the number of solutions of the congruence

$$(20) \qquad x_1^2+x_2^2\equiv t_1 \pmod{2^n}, \quad 0\leqq x_1,\ x_2<2^n.$$

Assume that $n\geqq 4$ and write[25] $x_1=y_1+2^{n-2}z_1$, $x_2=y_2+2^{n-2}z_2$, $0\leqq y_1$, $y_2<2^{n-1}$, $0\leqq z_1$, $z_2<2^2$. Then the residue classes of x_1 and x_2 (mod 2^n) are uniquely determined by those of y_1 and y_2 (mod 2^{n-2}) and those of z_1 and z_2 (mod 2^2). Also, (20) is equivalent to the pair of congruences:

$$(21) \qquad y_1^2+y_2^2\equiv t_1 \pmod{2^{n-1}}, \quad 0\leqq y_1,\ y_2<2^{n-2},$$

$$(22) \qquad y_1z_1+y_2z_2 \equiv \frac{t_1-y_1^2-y_2^2}{2^{n-1}} \pmod 2, \quad 0\leqq z_1,\ z_2<2^2,$$

since $2(n-2)\geqq n$ when $n\geqq 4$. The number of solutions of (21) is $M(2^{n-1},S_2,t_1)/4$. For, $y_1'\equiv y_1$ and $y_2'\equiv y_2$ (mod 2^{n-2}) imply that $y_1'^2\equiv y_1^2$ and $y_2'^2\equiv y_2^2$ (mod 2^{n-1}) so that (21) is satisfied by complete residue classes modulo 2^{n-2}, for both y_1 and y_2. The result claimed follows since in $M(2^{n-1},S_2,t_1)$ we have to count residue classes modulo 2^{n-1} and each residue class modulo 2^{n-2} contains exactly 2 of these. Moreover, all solutions of (21) are *primitive*, that is, at least one of y_1 and y_2 is prime to the modulus 2^{n-1}. Hence, in (22), we may assign the four values 0, 1, 2, and 3 modulo 2^2, arbitrarily to either z_1 or z_2 and then determine the other uniquely modulo 2, that is, in 2 ways modulo 2^2, to satisfy (22). Corresponding to each pair of solutions of (21) there are, therefore, 8 pairs of solutions of (22). This proves that, if $n\geqq 4$, then

$$M(2^n,S_2,t_1)=8(1/4)\ M(2^{n-1},S_2,t_1)=2M(2^{n-1},S_2,t_1).$$

By induction we obtain

$$M(2^n,S_2,t_1)=2^{n-3}M(2^3,S_2,t_1)$$

for $n\geqq 3$, and this proves (19). By trial, it can be verified that $M(2^3,S_2,t_1)$, t_1 odd, has the value 16 or 0 according as $t_1\equiv 1$ or 3 (mod 4). Hence, finally, $d_2(S_2,t_1)=2$ or 0 according as $t_1\equiv 1$ or 3 (mod 4).

[25] Cf. Landau, op. cit. 13), p. 283.

Jacobi's formula can now be obtained by inserting the values found for P, Q and $d_2(S_2,t_1)$ in (16) as follows: If $t_1 \equiv 1 \pmod 4$,

$$A(S_2,2^\alpha t_1) = A(S_2,t_1) = \frac{\pi}{2}\cdot 2\cdot\frac{4}{\pi}E(t_1) = 4E(t_1);$$

if $t_1 \equiv 3 \pmod 4$, $A(S_2,2^\alpha t_1) = 0 = 4E(t_1)$. It should be mentioned here that the different roles played in this problem by the primes $\equiv 1 \pmod 4$, the primes $\equiv 3 \pmod 4$, and the prime 2, correspond to the differences in their factorizations in the quadratic number field $R(i)$, in the connection mentioned in §2.

Case (*ii*). $S_4 = x^2+y^2+z^2+w^2$. If p is an odd prime, it is found that[26]

(23) $$d_p(S_4,t) = (1-1/p^2)(1+1/p+\cdots+(1/p)^{\alpha_p}),$$

where α_p is defined as in Case (i). On substituting in (14) we get

(24) $$A(S_4,t) = \pi^2\cdot t\cdot d_2(S_4,t)\cdot P_1\cdot Q_1(t),$$

where

$$P_1 = \Pi'(1-1/p^2),\ Q_1 = \Pi''(1+1/p+\cdots+(1/p)^{\alpha_p}).$$

For purposes of evaluation it is convenient to write $t = 2^\alpha t_1$, t_1 odd, $B = (1-1/2^2)P_1 = \prod\limits_{(p)}(1-1/p^2)$, and $C = t_1 Q_1(t)$. Then

(25) $$A(S_4,t) = \pi^2\cdot(4/3)\cdot 2\cdot d_2(S_4,t)\cdot B\cdot C.$$

The quantity C is the sum of the positive odd divisors of t, that is, of t_1, as can be seen by multiplying out and noting that $t_1 = \Pi' p^{\alpha_p}$. The infinite product B can be shown to be the reciprocal of the series

(26) $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6},$$

which Euler was able to evaluate after all efforts of James Bernoulli to do so had failed. Again Siegel's theorem provides an alternative arithmetical evaluation of (26) as was the case with (18). We substitute the value (26) for $1/B$ in (25) and obtain

(27) $$A(S_4,t) = 8\,[2^\alpha d_2(S_4,2^\alpha t_1)]C.$$

There remains only to prove that the quantity in brackets is 1 or 3 according as $\alpha = 0$ or $\alpha > 0$. This can be done by first proving that (19) holds with S_4 in place of S_2, t_1 odd, and then dividing the solutions

[26] Cf. Siegel, l. c. 22).

counted in $M(2^n, S_4, 2^\alpha t_1)$ into a set of primitive solutions and various sets of non-primitive solutions according to the highest power of 2 which is a common factor of the solutions. The latter sets correspond to sets of primitive solutions of congruences with lower powers of 2 as moduli. We omit the details.

Case (*iii*). $S_3 = x^2 + y^2 + z^2$. For an odd prime p which does not divide t it is found that[27]

$$d_p(S_4, t) = 1 + (-t|p)/p, \tag{28}$$

where $(-t|p)$ is again the Legendre symbol. For odd primes which divide t there are two expressions corresponding to (15) or (23) in the other cases, depending upon whether α_p is even or odd. As we do not propose to carry out the evaluation of $A(S_3, t)$ we omit the formulas. It suffices to say that these expressions, as well as the values (28), do not vanish, that is, $d_p(S_3, t) \neq 0$ for all odd primes p. Hence $d_2(S_3, t)$ is the critical quantity which determines whether or not $A(S_3, t)$ is 0. A necessary and sufficient condition that $d_2(S_3, t) = 0$ is that t be of the form $4^a(8b+7)$, as a congruencial study shows. It should also be noted here that the infinite product corresponding to P and B in the first two cases, respectively, in this case depends upon t. It is related to the class-number of binary forms of determinant $-t$. This relation between the representations of an integer by S_3 and the class-numbers of positive definite binary quadratic forms has been extensively studied.

We emphasize, in concluding this section, that the results we have obtained above from Siegel's theorems are very special, although the first proofs of them must properly be regarded as great mathematical achievements. It would be impossible in the space at our disposal to attempt to reproduce the proof of Siegel's fundamental theorem itself, or to indicate its further applications in the theory of automorphic functions, etc.

5. *Waring's Problem.* Of this problem, Professor Dickson has said: "It furnishes a typical example in the theory of numbers of the contrast between the ease with which empirical theorems are discovered and the difficulty attending a complete mathematical proof." Except in the special cases of fourth, fifth, and sixth powers, the difficulties attending the solution of Waring's problem have now been overcome. It is the purpose of this section to indicate briefly the nature of the results which have been obtained.

There is an infinitude of positive integers, including all integers $\equiv 3 \pmod 4$, which cannot be expressed as a sum of *two* integral squares

[27] Cf. Siegel, l. c. 22) *Hilfssatz* 10.

$\geqq 0$. Indeed, as we have seen, no positive integer which is $\equiv 7 \pmod 8$ is a sum of *three* such squares. On the other hand, *every* positive integer is expressible as a sum of *four* integral squares $\geqq 0$. The ordinary Waring problem is that of determining, for each integer $n \geqq 2$, the minimum number $g(n)$ such that every positive integer is a sum of $g(n)$ non-negative integral nth powers. Thus $g(2)=4$. It was conjectured by Waring[28] 1770 that $g(3)=9$, $g(4)=19$, and that a finite, but unspecified $g(n)$ exists for each n. About 1772, J. A. Euler, son of L. Euler, stated that

$$(29) \qquad g(n) \geqq I(n) = q + 2^n - 2,$$

where $q = [(3/2)^n]$, meaning that q is the greatest integer $\leqq (3/2)^n$. The number $I(n)$ is called the *ideal number* for nth powers, and the *ideal* Waring theorem states that $g(n) = I(n)$. Some values of $I(n)$ are 4, 9, 19, 37, 73, 143, and 279 for $n=2, 3, 4, 5, 6, 7$, and 8, respectively, while $I(n) > 2000000$ when $n \geqq 21$. The inequality (29) becomes obvious when one observes that the integer $2^n q - 1$ is the sum of $(2^n - 1)$ nth powers each $=1$ and $(q-1)$ nth powers each $=2^n$, and that this integer is not a sum of fewer than $(2^n-1)+(q-1)=I(n)$ non-negative integral nth powers. It is by no means a trivial problem to prove, for each n, that a finite $g(n)$ exists, that is, to prove that for a fixed n, the number of nth powers required is bounded for all positive integers. There is now, however, an entirely algebraic proof of this theorem, first obtained by Hilbert in 1909 through the study of a certain multiple integral.[29]

That $g(3) = I(3) = 9$ was first proved by Wieferich and Kempner. Landau proved in 1909 that every integer exceeding a fixed value is expressible as a sum of *eight* non-negative integral cubes. The analysis shows that, in general, when $n > 6$, only a finite number of positive integers actually require $I(n)$ nth powers, for example, $g(12) = 4223$, while all integers $\geqq 2.3^{12}$ can be expressed as the sum of 2405 non-negative integral 12th powers. In contrast with the requirements for integral cubes it may be mentioned that every integer is expressible as a sum of *three* rational cubes $\gtreqless 0$.

Hilbert's proof of the finiteness of $g(n)$ and the algebraic modifications of his proof depend upon the proof of the existence of certain algebraic identities. By means of such identities in the special case $n=4$ it was successively shown by Liouville, Lucas, Fleck, Landau and Wieferich that $g(4) \leqq 53$, 41, 39, 38 and 37, respectively. Dick-

[28] Waring, *Meditationes algebraicæ* (Cambridge, 1770).

[29] Hilbert, Göttingen Nachrichten (1909), pp. 17-36; Mathematische Annalen, 67 (1909), pp. 281-300. For other references see Dickson, *History of the Theory of Numbers*, Vol. II, Chapter XXV.

son[30] has proved that every integer is expressible as a sum of 37 biquadrates of which 20 may be taken equal in pairs, that is to say, only 27 variables are required. It is not yet known whether or not $g(4)=I(4)$.

The analytical theory of Hardy and Littlewood[31] yields the result that every sufficiently large integer is expressible as a sum of

$$(n-2)2^{n-1}+5 \tag{30}$$

non-negative integral nth powers if $n \geqq 3$. Their theory yields also asymptotic formulas for the number of representations. In comparison with the more recent results[32] of Winogradow, Dickson and Pillai, (30) is very large except in the cases $n=4$, 5 and 6, in which it is better than these later results. Moreover, the evaluation of a constant beyond which the number (30) of nth powers suffices was very difficult, and exceedingly large values for such constants were obtained. It is the combination of the new analytic theory originating with Winogradow and simplified by Dickson, and, independently by Pillai, with Dickson's algebraic method of ascent, which has led to the solution of Waring's problem for $n>6$. The principal theorems which describe the results are Theorems 1 and 2 which follow.

Theorem 1. *Let* $3^n=2^nq+r$, $0 \leqq r < 2^n$. *If* $n>6$, and

$$2^n \geqq q+r+3, \tag{31}$$

then $g(n)=I(n)$; *that is, the ideal Waring theorem holds.*

The inequality (31) holds if $4 \leqq n \leqq 400$. If it should fail for any $n>400$, then the decimal for $r/2^n$ would begin with at least 50 digits equal to 9.

Theorem 2. *If* $n>6$ *and* (31) *fails, write* $f=(4/3)^n$. *Then* $g(n)=I(n)+f$ *or* $g(n)=I(n)+f-1$ *according as* $2^n=fq+f+q$ *or* $2^n<fq+f+q$.

[30] Cf. Dickson, l. c. 11), pp. 255-261.

[31] Landau has given an exposition of the work of Hardy and Littlewood, op. cit. 13), pp. 235-360.

[32] Dickson gave a report on Waring's problem, Bulletin of the American Mathematical Society, 39 (1933), pp. 701-727. Among the most important later papers are: Winogradow, Annals of Mathematics, 36 (1935), pp. 395-405. Dickson, ibid., 37 (1936), pp. 293-316.
——, American Journal of Mathematics, 58 (1936), pp. 521-535. Pillai, Journal of the Indian Mathematical Society (new series), 2 (1936), pp. 16-44.
Dickson, Acta Arithmetica, 2 (1937), pp. 177-196.
The results in the last paper were announced in the Bulletin of the American Mathematical Society, March 13, 1936, p. 341. Theorems 1 and 2 above, and the remarks following them, are taken from Dickson, Bulletin of the American Mathematical Society, 42 (1936), pp. 833-842. This paper was delivered in an address by Professor Dickson at The Tercentenary Conference of Arts and Sciences at Harvard University, September, 1936.

Dickson's result above concerning the case $n=4$ suggests the possibility in general of reducing materially the number of variables required. For example, Dickson has shown that, if $9 \leqq n \leqq 400$, in the ideal Waring theorem one may take $2P$ of the powers equal in pairs, where $P=(2^n+q-4n+d)/2-2$, where $d=1$ or 2 according as q is odd or even. This reduces the number of variables to $I-P$.

Generalizations of Waring's problem to the case of polynomial summands have been made, and many of the problems solved by essentially the same methods. In the case of homogeneous polynomial summands in 2 variables other difficulties arise.

ERRATA

Page 199, 9th line from top of page. omit period and use a lower case t following, "at the time the State Teachers' Association was in session."

Page 206, 20th line from top after G. Baron use a period instead of a colon.

Page 208, 19th line bottom omit "our" read "list of numerical authors."

Page 208, 8th line from bottom, for "Mangham" read "Manghan."

Page 209, 7th line from top for "Mangham" read "Manghan".

Page 209, 11th line from top for "Mangham" read "Manghan."

Page 209, 25th line from top for "notations" read "notation".

MATHEMATICKLERS

An examination question asking for the statement of the fundamental theorem of the integral calculus was productive of the usual amount of mangled English and mathematics. One student, however, gave a practically perfect answer. He described accurately how one formed the sum

$$\sum_{i=1}^{\infty} f(x_i)\Delta_i x$$

for a given subdivision of the interval of integration, and indicated how the process was to be carried out repeatedly for finer and finer subdivisions of the interval. Then he wrote, "Now let n increase without rebound."

A Note on Integration

By Carl A. Ludeke
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Since the advent of the quantum theory and the use of wave packets involving the Gaussian error function and the Coulomb potential function, the applied mathematician is often faced with integrals which can be reduced to the form:

$$(1)\quad I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-(1/\sigma_1^2)(x_1^2+y_1^2+z_1^2)-(1/\sigma_2^2)(x_2^2+y_2^2+z_2^2)\right.$$
$$\left.+f(x_1-x_2,\ y_1-y_2,\ z_1-z_2)\right]dv_1dv_2/\left[(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right]^{1/2}$$

where $dv_1=dx_1dy_1dz_1$, $dv_2=dx_2dy_2dz_2$, and the form of $f(x_1-x_2,\ y_1-y_2,\ z_1-z_2)$ depends on the nature of the physical problem.

Letting $X=x_1-x_2$, $Y=y_1-y_2$, $Z=z_1-z_2$, equation (1) becomes:

$$(2)\quad I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-(1/\sigma_1^2)\left[(x_2+X)^2+(y_2+Y)^2+(z_2+Z)^2\right]\right.$$
$$\left.-(1/\sigma_2^2)(x_2^2+y_2^2+z_2^2)+f(X,Y,Z)\right\}dv_2dV/(X^2+Y^2+Z^2)^{1/2}$$

where $dV=dXdYdZ$. Regardless of the form of $f(X,Y,Z)$, the three integrations over dv_2 can be carried out immediately by writing equation (2) in the form:

$$(3)\quad I=\int_{-\infty}^{\infty}\exp\left[-(1/\sigma^2)(X^2+Y^2+Z^2)\right.$$
$$\left.+f(X,Y,Z)\right]dV/(X^2+Y^2+Z^2)^{1/2}\int_{-\infty}^{\infty}\exp\left\{-(\sigma^2/\sigma_1^2\sigma_2^2)\left[\left(x_2+\frac{\sigma_2^2}{\sigma^2}X\right)^2\right.\right.$$
$$\left.\left.+\left(y_2+\frac{\sigma_2^2}{\sigma^2}Y\right)^2+\left(z_2\frac{\sigma_2^2}{\sigma^2}Z\right)^2\right]\right\}dv_2$$

where $\sigma^2=\sigma_1^2+\sigma_2^2$. For now the integrations over dv_2 can be evaluated by the use of the formula:

$$(4)\qquad \int_{-\infty}^{\infty}\exp\left\{-p^2\left[(x-a)^2+(y-b)^2+(z-c)^2\right]dv=\pi^{3/2}/p^3\right.$$

where $a,b,c,$ and p are constants.

Equation (3) then becomes:

$$(5)\quad I=(\pi^{3/2}\sigma_1^3\sigma_2^3/\sigma^3)\int_{-\infty}^{\infty}\exp\left[-(1/\sigma^2)(X^2+Y^2+Z^2)+f(X,Y,Z)\right]dv/(X^2+Y^2+Z^2)^{1/2}.$$

Thus three of the six integrations in equation (1) have been evaluated without regard to the form of $f(X,Y,Z)$. The evaluation of the remaining three integrations depends, of course, on the form of $f(X,Y,Z)$ and hence on the nature of the physical problem.

As an example of this method, consider the evaluation of the Coulomb energy of two free electrons.* The energy equation is:

$$(6)\quad V_{11}=e^2N^4\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\exp\{-(1/\sigma^2)[x^2+y^2+(z-r_0)^2+(x+X)^2+(y+Y)^2+(z+Z)^2]\}dvdV/(X^2+Y^2+Z^2)^{1/2}$$

where V_{11} is the Coulomb energy of two free electrons
r_0 is the distance between the electrons.
e is the charge on the electrons.
N is the normalizing factor $=(1/\pi^{3/4}\sigma^{3/2})$.

According to the method of this article, equation (6) is written in the form:

$$(7)\quad V_{11}=e^2N^4\int_{-\infty}^{\infty}\frac{\exp\left[-(\frac{1}{2}\sigma^2)(X^2+Y^2+Z^2+r_0^2+2r_0Z)\right]dV}{(X^2+Y^2+Z^2)^{1/2}}\cdot\int_{-\infty}^{\infty}\exp\left[-(2/\sigma^2)\{(x+\tfrac{1}{2}X)^2+(y+\tfrac{1}{2}Y)^2+[z+\tfrac{1}{2}(Z-r_0)]^2\}\right]dv.$$

The integrations over dv are carried out with the help of formula (4).

Thus equation (7) becomes:

$$V_{11}=e^2\exp(-r_0^2/2\sigma^2)/2^{3/2}\pi^{3/2}\sigma^3\cdot\int_{-\infty}^{\infty}\exp\left[-(\tfrac{1}{2}\sigma^2)(X^2+Y^2+Z^2+2r_0Z)\right]dV/(X^2+Y^2+Z^2)^{1/2}.$$

*C. A. Ludeke, *On the Interchange Energy of Two Free Electrons*, The Physical Review, v. 55, p. 315.

Upon changing to polar coordinates and making use of the formula:*

$$\int_0^\infty \exp(pg+\beta p^2)dp = (\pi^{1/2}/2i\beta^{1/2})\exp(-g^2/4\beta)[1-\mathrm{erf}(ig/2\beta^{1/2})]$$

where $$\mathrm{erf}(z) = \tfrac{1}{2}\pi^{1/2}\int_0^z \exp(-t^2)dt \quad \text{and} \quad R(\beta)<0,$$

we find:

$$V_{11} = (e^2/r_0)\mathrm{erf}(r_0/2^{1/2}\sigma).$$

In this example $f(X,Y,Z)$ is zero. However, in the expression for the interchange energy of two free electrons $f(X,Y,Z)$ is not zero. Nevertheless, it is possible to carry out all the integrations after having removed three of them by the above method.†

*Bell Telephone System, Monograph B-584, September, 1931, Table 1, No. 1317.

†C. A. Ludeke, *On the Interchange Energy of Two Free Electrons*, The Physical Review, v. 55, p. 315.

MATHEMATICKLERS

In a classroom discussion of infinite geometric progressions, the instructor presented the example of the frog jumping across the road in such a fashion that at each jump he covered half of the remaining distance to the edge of the road. After some discussion the instructor posed the question of whether the frog would ever get across the road. Further discussion convinced most of the class that it depended on how fast the frog jumped. In particular, if the frog covered half of the distance in half a minute, half of the remaining distance in a quarter of a minute, etc., it was generally agreed that the elapsed time for all of the jumps could not exceed one minute, so that after one minute the crossing of the road would necessarily be completed. One student protested vigorously. "The frog", said he, "would obviously never get across the road, because he would never get his minute used up."

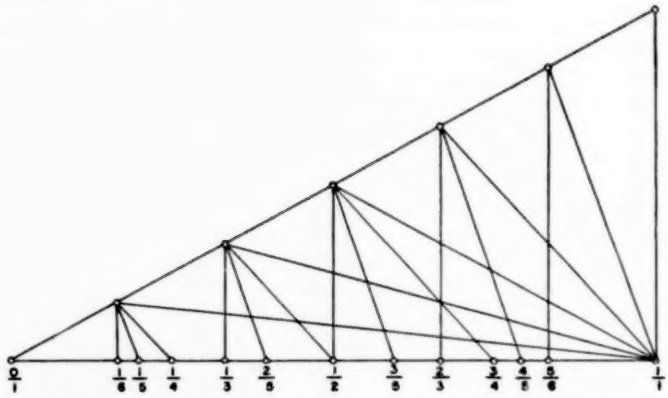
Additive Entities, An Extension of Farey Series

By Harold Sinclair Grant
Rutgers University

A Farey Series of order n consists of all rational numbers a/b, $(a,b)=1$, $0<b\leqq n$, written down in order of magnitude, (a,b) representing the greatest common divisor of a and b. Without loss of generality, we can evidently restrict ourselves to an interval between two successive integers, g and $g+1$, say, so that $g\leqq a/b\leqq g+1$. For the fractions in the preceding interval, $g-1\leqq a/b\leqq g$, are obtained by subtracting 1, and those in the succeeding interval, $g+1\leqq a/b\leqq g+2$, are obtained by adding 1. We restrict ourselves to the interval $0\leqq a/b\leqq 1$, so that the Farey Series of order 6 is:

$$0/1,\ 1/6,\ 1/5,\ 1/4,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 3/4,\ 4/5,\ 5/6,\ 1/1.$$

There is evidently a one-to-one reciprocal correspondence between the terms of a Farey Series and the points of rational sub-division of a linear segment in Euclidean space, to n sub-divisions. This is illustrated below for $n=6$.



A Farey Series of order n exhibits the following three properties:

(1) If a/b, c/d are successive terms, then $ad-bc=-1$.

(2) If a/b, c/d are successive terms, then $b+d\geqq n+1$.

(3) If a/b, e/f, c/d are three successive terms, then $e/f=a+c/b+d$.

The third property, that actually stated by Farey, is an immediate consequence of the first, so that its importance may be readily overlooked. Many proofs have been given establishing the first property.*

In this short paper, we wish to emphasize the third property, to show that it is the "generating" property of a Farey Series, and that properties (1) and (2) follow from it. The fact that the third property enables us to write down immediately the succssive terms of any Farey Series should be sufficient to establish its importance. It is not at all surprising that these Series have found an application in Additive Arithmetic when one considers that they may be "generated" by the most natural operation of arithmetic, addition. Since our subsequent discussion will not be confined to Farey Series alone, but will be made as general as possible, we call attention to the fact that the first two properties of Farey Series are essentially those of the component integers, and not of the fractions themselves. Thus, in starting with the third property to generate our series, we may remove the idea of fraction entirely, and regard a Farey Series of order n as the set of number pairs $\{a,b\}$, $(a,b)=1$, $0<b\leqq n$, $0\leqq a\leqq b$, in which $\{a,b\}$ is written to the left or right of $\{c,d\}$ according as $ad<bc$ or $ad>bc$.

We start with what we are pleased to call two additive entities, a left-hand entity A and a right-hand entity B. We will assume that the operation of addition is defined for A and B and that:

(1) It is associative, that is, e. g. $A+(A+B)=(A+A)+B$.

(2) It is commutative, that is, e. g. $A+B=B+A$.

(3) For either A or B, $nA+A=A+nA=(n+1)A$, for non-negative, integral n.

Many examples of such entities could be given; one or two will suffice. A and B might be real or complex numbers; they might be vectors; they might be angles; or they might be matrices. We build up an infinite system of such entities according to the following rule: To form an entity intermediate to any two, we add. Thus, starting with A on the extreme left and B on the extreme right, we obtain first $A+B$, then $2A+B$ and $A+2B$, etc., thus:

$$A,\ 3A+B,\ 2A+B,\ 3A+2B,\ A+B,\ 2A+3B,\ A+2B,\ A+3B,\ B.$$

We write $rA+sB=[r,s]$, and note the following facts:

(1) r and s are non-negative integers.

*See, for example, E. Landau's *Vorlesungen über Zahlentheorie*, V. 1, pp. 98-100. G. H. Hardy outlines an interesting geometric proof in his excellent lecture *An Introduction to the Theory of Numbers*, Bulletin of the American Mathematical Society, Vol. 35, No. 6, November-December, 1929.

(2) If $[r_1,s_1]$, $[r_2,s_2]$ are two adjacent entities at any stage, then $[r_1+r_2,\ s_1+s_2]$ is the intermediate entity at the next stage.

(3) The extreme entities are $[1,0]$ and $[0,1]$, and are the only entities $[r,s]$ in which either r or s is zero.

(4) No two entities are identical, since 0 is the only integer for which $n+0=0+n=n$.

(5) To every entity $[r,s]$ $r>s$, on the left of $[1,1]$, there corresponds an entity $[s,r]$ on the right of $[1,1]$, symmetrically situated.

(6) If $r+s\leqq n$ and $[r_1,s_1]$, $[r_2,s_2]$ are adjacent at this stage, then $r_1+r_2+s_1+s_2\geqq n+1$. This is true for $n=1$. If it is true for $n=m$, then it must be true for $n=m+1$. For, if $r_1+r_2+s_1+s_2\geqq m+1$, then either $r_1+r_2+s_1+s_2\geqq m+2$, in which case no intermediate entity to $[r_1,s_1]$, $[r_2,s_2]$ occurs at the stage $n=m+1$, or $r_1+r_2+s_1+s_2=m+1$, in which case the intermediate entity $[r_1+r_2,\ s_1+s_2]$ is formed, and we have:

$$r_1+r_1+r_2+s_1+s_1+s_2=r_1+s_1+m+1\geqq m+2$$

and

$$r_1+r_2+r_2+s_1+s_2+s_2=r_2+s_2+m+1\geqq m+2.$$

(7) A necessary and sufficient condition that two entities $[r_1,s_1]$, $[r_2,s_2]$ be adjacent at some stage is that $r_1s_2-r_2s_1=1$. The necessity follows from the fact that $r_1s_2-r_2s_1$ is an invariant of our "building up" process, and is therefore the same as for the two extreme entities $[1,0]$ and $[0,1]$. For, considering the entities $[r_1,s_1]$, $[r_1+r_2,\ s_1+s_2]$, $[r_2,s_2]$, we have:

$$r_1(s_1+s_2)-s_1(r_1+r_2)=(r_1+r_2)s_2-(s_1+s_2)r_2=r_1s_2-s_1r_2.$$

An immediate consequence of this result is that in any entity $[r,s]$, $(r,s)=1$.

To prove that $r_1s_2-r_2s_1=1$ is a sufficient condition that the entities $[r_1,s_1]$ and $[r_2,s_2]$ be adjacent at some stage, we will show that $r_1s_2-r_2s_1$ increases with the number of entities that originally separate $[r_1,s_1]$ and $[r_2,s_2]$. Suppose we have $[r_1,s_1]$, $[r_2,s_2]$, $[r_3,s_3]$, where $r_2\neq r_1+r_3$, $s_2\neq s_1+s_3$, so that $[r_1,s_1]$ and $[r_3,s_3]$ are never adjacent. Then:

$$r_1s_2-s_1r_2=1$$

$$-r_3s_2+s_3r_2=1,$$

whence:

$$s_2=\frac{\begin{vmatrix} 1 & -s_1 \\ 1 & s_3 \end{vmatrix}}{\begin{vmatrix} r_1 & -s_1 \\ -r_3 & s_3 \end{vmatrix}}=\frac{s_1+s_3}{r_1s_3-r_3s_1}\ ; \quad r_2=\frac{r_1+r_3}{r_1s_3-r_3s_1}\ .$$

From this it follows that $r_1s_3-r_3s_1$ is positive and therefore greater than 1. Suppose now we have: $[r_1,s_1],[r_2,s_2],\cdots\cdots[r_{m-1},s_{m-1}],[r_m,s_m]$, $m\geqq 3$, where $[r_1,s_1]$ and $[r_m,s_m]$ are separated by at least $(m-2)$ entities. We will show by induction that $r_1s_m-r_ms_1>r_1s_{m-1}-r_{m-1}s_1$. This inequality is true for $m=3$, from above. Suppose it true for $3\leqq m\leqq n$, then we wish to show that $r_1s_{n+1}-r_{n+1}s_1>r_1s_n-r_ns_1$. Now:

$$r_ns_{n+1}-r_{n+1}s_n=1$$

$$r_{n-1}s_{n+1}-r_{n+1}s_{n-1}=k>1,$$

whence: $$s_{n+1}=\frac{\begin{vmatrix}1 & -s_n\\ k & -s_{n-1}\end{vmatrix}}{\begin{vmatrix}r_n & -s_n\\ r_{n-1} & -s_{n-1}\end{vmatrix}}=\frac{ks_n-s_{n-1}}{r_{n-1}-r_ns_{n-1}}=ks_n-s_{n-1};$$

$$r_{n+1}=kr_n-r_{n-1}.$$

Therefore: $$\begin{aligned}r_1s_{n+1}-r_{n+1}s_1&=r_1(ks_n-s_{n-1})-s_1(kr_n-r_{n-1})\\&=r_1ks_n-r_1s_{n-1}-s_1kr_n+s_1r_{n-1}\end{aligned}$$

$$=(r_1s_n-r_ns_1)k-(r_1s_{n-1}-r_{n-1}s_1)>(r_1s_n-r_ns_1)k-(r_1s_n-r_ns_1)$$
$$=(r_1s_n-r_ns_1)(k-1)\geqq r_1s_n-r_ns_1.$$

(8) If $r+s\leqq n$, r and s run through all pairs of non-negative integers, and only such pairs, for which $(r,s)=1$. This is true for $n=1$. We wish to show that if it holds true for $n=m$, it will hold true for $n=m+1$. In other words, there are two and only two adjacent entities, say $[p_1,q_1],[p_2,q_2]$ at the stage $n=m$, which will give rise at the next stage to each entity $[r,s]$, where $r+s=m+1$, $(r,s)=1$. Now we must have: $[p_1,q_1],[r,s],[p_2,q_2]$, with the four independent equations: $p_1q_2-p_2q_1=1$, to insure adjacency; $p_1+p_2=r$; $q_1+q_2=s$; $p_1+q_1=a$, where $0<a\leqq m$. Eliminating p_2 and q_2 from the first equation, we obtain: $p_1s-q_1r=1$, from which $p_1s-(a-p_1)r=1$, or

$$p_1=\frac{ar+1}{r+s}=\frac{ar+1}{m+1}\quad.$$

As a varies from 1 to $m+1$, ar, and therefore $ar+1$, goes through a complete residue system, mod $(m+1)$. Therefore a is determined uniquely, since, for $a=m+1$, the remainder is obviously 1. It is readily verified that:

$$q_1=\frac{sa-1}{m+1}\ ;\quad p_2=\frac{(m+1-a)r-1}{m+1}\ ;\quad q_2=\frac{(m+1-a)s+1}{m+1}\quad.$$

If now we take $A = \{0,1\}$, $B = \{1,1\}$, and define the operation of addition by: $\{a,b\} + \{c,d\} = \{a+c,b+d\}$, we have:

$$[r,s] = rA + sB = \{0,r\} + \{s,s\} = \{s,r+s\}.$$

We note that: $a(b+d) < b(a+c)$ and $(a+c)d < (b+d)c$, when $ad < bc$. Further, if $r+s \leqq n$, r and s run through all pairs of non-negative integers, and only such pairs, for which $(r,s) = 1$. Now $(r,s) = 1$ when, and only when, $(r+s,s) = 1$. Hence s will run through all non-negative integers $\leqq m$ such that $(s,m) = 1$, $0 < m \leqq n$, and only such integers. Restricting ourselves to the interval between 0 and 1, this is precisely the way we form the Farey Series of order n. Suppose, now, $\{s_1,r_1+s_1\}$, $\{s_2,r_2+s_2\}$ are two successive terms in our series of order n. We have: $s_1(r_2+s_2) - s_2(r_1+s_1) = s_1r_2 - s_2r_1 = -1$ (property (7)). Also: $r_1+s_1+r_2+s_2 \geqq n+1$ (property (6)). The immediate writing down of the Farey Series of order (6) is illustrated below, the numbers above indicating the order in which the terms were written down:

$$\begin{array}{cccccccccc} 1 & 12 & 10 & 6 & 4 & 7 & 3 & 8 & 5 & 9 \\ \{0,1\} & \{1,6\} & \{1,5\} & \{1,4\} & \{1,3\} & \{2,5\} & \{1,2\} & \{3,5\} & \{2,3\} & \{3,4\} \end{array}$$

$$\begin{array}{ccc} 11 & 13 & 2 \\ \{4,5\} & \{5,6\} & \{1,1\} \end{array}$$

MATHEMATICKLERS

As evidence that not all ludicrous boners are made by college students, witness the following startling definition which appears in a certain textbook on a more or less advanced mathematical subject: "An asymptotic series is an infinite series which converges for a certain number of terms, and then begins to diverge."

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from January issue.)

Number 6, Vol. I, contains the solutions of the problems proposed in No. 5, ***An Essay on the First Elements of Fluxions***, being the first section of the ***Principles of Fluxions*** by Rev. S. Vince, A. M., F. R. S., Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge, England. Edition, 1800. Republished at the request of Walter Folger, Jr., Nantucket; solutions of problems in No. V and concludes with ten new problems proposed for solution.

Number 7, Vol. I, contains the solutions of the problems proposed in No. 6; a continuation of the ***Essay on Fluxions*** and eleven problems proposed for solution.

Question 53, the second of the list published in No. 6 was proposed by A. Rabbit and reads as follows:

Required a more elegant theorem than that given by Mr. Shepherd on page 203 of his *Columbian Accountant*, for finding the side of what he calls the greatest inscribed square of an equilateral triangle.

In his solution of the problem, A. Rabbit says, among other things, "The contents of page 203 of the *Columbian Accountant* sung to the tune of 'the pump* of Lake Champlain' will completely exhibit the stupidity of Shepherd's rule mentioned in the question. With this remark I proceed to develop the theorem sought."

The star refers to the following foot-note: "The Theological properties of this wonderful pump have, I understand, been sufficiently investigated by the members of a certain church in New York; and I promise to unfold the mathematical principles of the same, in some future Number. A. R."

On page 154 of this number, under "acknowledgments", the editors say, "A. Rabbit will not, in any future number, be permitted

to propose questions concerning the blunders of stupid Shepherd: We had rather soar aloft with the eagle than waddle in the mud with the goose." Also this: "some of our country subscribers have discovered that the best way of encouraging the mathematics is never to remit us a single cent for our numbers. This new idea is honorable to those who discovered it; posterity would certainly thank us were we to publish their names."

Number 7 is the first number we were able to find which contained a cover, Nos. 7, 8, and the double number 9 and 10, all having covers, are in the possession of Dr. Artemas Martin, of Washington, D. C. The outside of the front cover of number 7 contains the following:

THE MATHEMATICAL CORRESPONDENT:

Containing New Elucidations, Discoveries, and Improvements in the Various Branches of the Mathematics: with collections of Mathematical Questions resolved by ingenious correspondents.

Price to subscribers, 18¾c., to non-subscribers, 25c. each
N. B. No. 8 will be published on the first of May next.

RULES

to be observed by the contributors to the *Mathematical Correspondent.*

1. All communications must be post paid, and directed to "The Editors of the *Mathematical Correspondent*, New York. Communications from New York and its vicinity, by private conveyance must be left at Mr. Baron's, No. 24, Cedar Street.

2. No person will be permitted to propose in the work a question which he cannot resolve, and consequently those who wish to have questions inserted must send a true solution along with each question.

3. Communications for No. 8 must arrive before the twenty-fourth of January next, otherwise they cannot be inserted in that number.

☞Contributors to the *Mathematical Correspondent* are requested to give the solutions of questions at full length; the mere answer to a question cannot in future be acknowledge.

NEW YORK:

PRINTED FOR THE EDITORS.

Sold by T. and J. Swords, No. 160 Pearl-Street; W. P. Ferrand, Philadelphia; Dr. Mackay, No. 2 George-Street, Great-Tower-Hill, London, 1805.

On the inside of the front cover is a list of Mathematical publications with prices, for sale by T. and J. Swords. We note a few of the most important.

Bonnycastle's *Geometry*	$ 2.00
Hutton's *Mathematics*	7.00
Hutton's *Mensuration*	5.75
Maclaurin's *Fluxions*, 2 Vols.	10.00
Philosophical Magazine, 11 Vols.	43.50
Walker's *Conic Sections*, 4to.	6.50
Ward's *Mathematics*	2.25
Emerson's *Miscellanies*	3.00
Emerson's *Astronomy*	3.00
Emerson's *Principles of Mechanics*	7.50
Enfield's *Philosophy*	8.00

On the inside of the back cover is an advertisement by the same firm for the sale of the Medical Repository and the Domestic Encyclopedia, together with Globes, Maps, Charts, Gunter's scales, Mathematical Instruments in cases, etc. On the outside of the back cover is the following:

Navigation and the Lunar Observations, the whole circle of the mathematics, Analytics, Astronomy, and the Mathematical Principles of Natural Philosophy taught by G. Baron, No. 24 Cedar-Street, New York. N. B. An evening school from the first of November to the first of May.

AN ENIGMA

What is it that is neither animal, vegetable, nor mineral; of neither sex, yet between both; given from four to six feet; recorded in the Old Testament and strongly recommended in the New?

A TRUE PROPOSITION.

Four trees may be so planted that each tree shall be equally distant from every one of the three remaining trees. Pray how can this be done?

Number 8, Vol. I contains the solutions of the problems proposed in No. 7, an article entitled, *The Two First Chapters of Baron Masère's Dissertation on the use of the Negative Sign in Algebra, published with a view of drawing the attention of American Mathematicians to a careful perusal of the work,* and ten problems proposed for solution.

On page 174, under "Acknowledgements" it is stated, "The health of Mr. Baron, our principal editor, was last summer entirely destroyed

by three of the understrappers of the *Health Committee*. Deprived of his assistance, we earnestly solicit our contributors to render their solutions as perfect as possible."

It was here contemplated that the first volume of the *Correspondent* should contain $10\frac{1}{2}$ numbers.

The front cover of No. 8 is the same as that of No. 7, except no announcement is made as to the time when No. 9 will appear. It bears the date, 1806.

The inside of the front cover contains the same list of books for sale as is on the inside cover of No. 7. The inside of the back cover contains the following notice:

TO MATHEMATICIANS.

A bad state of health has induced me to offer for sale, a few of my books at the annexed prices. Those printed in Italics are exceedingly scarce. Immediate orders with ready money sent, post paid, to myself, will be duly executed.—G. BARON.

1.	Hayes' *Fluxions*, fol.	$ 6.00
2.	Kersey's *Algebra*, 2 Vols. fol.	12.00
3.	Sanderson's *Algebra*, 2 Vols. 4to	12.00
4.	Hutton's *Mensuration*, 4to	6.00
5.	Hodgson's *Fluxions*, 4to	6.00
6.	*Optics, Enumeratio Linearum*, etc., by Sir Isaac Newton, 1 Vol. 4to	8.00
7.	Malcom's *Arithmetic* 4to	20.00
8.	Emerson's *Mechanics*, 4to.	7.50
9.	Emerson's *Increments*, 4to.	3.50
10.	*Analysis per quantitatum Series*, etc., by Sir Isaac Newton, 1 Vol. 4to	5.00
11.	*A new and elegant Atlas*, published by Laurie and Whittle, 1802	10.00
12.	*The Philosophical and Mathematical Commentaries of Proclus on Euclid*, etc., translated by Taylor, 2 Vols. 4to.	8.00
13.	*Mathematical Essays* by Hellins, 4to.	2.50
14.	*Encyclopedia Britannica*, Edinburgh edition, 1802, 20 Vols. 4to., elegantly bound	200.00
15.	*The Philosophical Transaction of the Royal Society of Edinburgh*, Vols. 1, 2, and 3	20.00
16.	*Philosophical Transactions of the Royal Irish Academy*, 4 Vols.	20.00
17.	Taylor's *Tables*, neatly bound	30.00

18.	Rowe's *Fluxions*, 8vo.	2.50
19.	Sander's *Fluxions*, 8vo.	3.00
	(outside of back cover)	
20.	Simpson's *Fluxions*, 2 Vols. 8vo.	5.00
21.	Holiday's *Fluxions*, 8vo.	3.50
22.	Walfin's *Algebra*, 8vo.	3.00
23.	Barrow's *Mathematical Lectures*, 8vo.	3.00
24.	Barrow's *Geometrical Lectures*, 8vo..	3.00
25.	Newton's *Algebra*, 8vo.	2.00
26.	Keill's *Astronomical Lectures*, 8vo..	2.50
27.	Hutton's *Diaries*, 6 Vols. 12mo	6.00
28.	Nicholson's *Natural Philosophy*	3.00
29.	Keill's *Natural Philosophy*.	2.50
	One pair of Adams' 18 in. Globes	60.00
	One pair of new 12 in. British Globes with red leather cover	35.00

TO SUBSCRIBERS.

The Indisposition of Mr. Baron has unavoidably delayed the publication of this number. Several of our readers are anxious to know whether the publication of this work will cease at the end of the first volume. We inform them that we are determined to commence a second volume with redoubled vigor.—EDITORS M. C.

MARCH 10TH, 1806.

Then follows an advertisement of T & J. Swords.

Numbers 9, 10, etc., November 18th, 1806, Vol. I contains the solutions of the problems in No. 8, continuation of The First Ten Chapters of Baron Masère's *Dissertation on the Negative Sign in Algebra*, and an article entitled, *A View of Diophantine Algebra*, by R. Adrain, and a list of the subscribers, 347 in all. Some of these paid for as many as six subscriptions.

On page 201, it is stated that Mr. Baron owing to other engagements cannot possibly be concerned in editing the future numbers of the *Mathematical Correspondent*. This number contains a portrait of Mr. Baron painted by the *Celebrated Dr. Anderson of New York*. This portrait is not in the bound copy of Vol. I, which is in the Lenox Library of New York, but is in a duplicate number of 9, 10, etc., which is bound in a volume with other pamphlets and labeled *Scientific Pamphlets*. Dr. Artemas Martin of Washington, D. C. possesses a bound copy of Vol. I, which does contain the portrait.

It is also stated in this number that "the first number of Vol. II will appear about the first of May next."

The article on Diophantine Algebra in this number is believed to be the first of the kind published in America.

The outside of the front cover of this number is the same as that of No. 8, with the exception that no announcement is made as to the probable time of issuing the next number. In consequence of this number being a double number, we infer that the price to subscribers was raised to 50c. and to non-subscribers 62½c.

Also the following appears on the cover:

TO SUBSCRIBERS.

GENTLEMEN:

Lately an ill state of health from which I have but just recovered, combined with many domestic evils, resulting from the diabolical machinations of a parcel of illiterate villains with whom I have the misfortune to be allied, have nearly consigned me to the grave. Surrounded with these evils, which, for the sake of humanity, I hope seldom exists, I was unable to write, read, or even to think on scientific subjects; and of consequence the publication of the *Mathematical Correspondent* has been delayed from the first day of June last to the present. Curiosity, I know will ask me to whisper some explanation, but I tell her, my independent mind, despising secret whispers, will soon employ my pen in delineating a correct portrait* of these villains, to be held up for public execration of the scientific world. To you worthy subscribers, my fellow editors, and myself tender the thanks of our most grateful hearts. The patrons of Science are the friends of mankind. You have had the honor of supporting the first work of this nature ever attempted in the United States, a work which has received the approbation of the learned both at home and abroad. May our feeble labours founded on your generous assistance, prove a public benefit to the youths of our country.—Subscribers indebted to the editors will please to forward me the same they severally owe. This is absolutely necessary; for an immediate payment must be made of all debts contracted for the execution of this work. On the last two pages of this cover, you will perceive that I intend to commence the publication of a large work, which to seamen and others is, perhaps, one of the most happy discoveries in modern times. For this reason alone it is that I cannot assist in editing the future numbers of the *Mathematical Correspondent*.

New York, November 18th, 1806. G. BARON.

The above is followed by an advertisement of a Plane Table, by T. and J. Swords.

On the inside and outside of the back cover, Mr. Baron expatiates on what he terms *A New and Important Discovery*.

This explanation we give in full as it gives Mr. Baron's reasons for withdrawing his connection with the *Correspondent*,—a reason apparently not generally known.* He says,

"To the Friends of Arts and Sciences in America,

GENTLEMEN:

I beg leave to draw your attention to the nature of a work for which I intend a short time hence to solicit your patronage by subscription. In our present state of

*These to our subscribers will be sent gratis.

existence every man has some favorite pursuit peculiar to himself. The Mathematics, Astronomy, and Philosophy, have from my earliest youth to the present time engrossed all my attention; and in each of these subjects, I have unexpectedly, gained some degree of celebrity. Of all the arts founded on Mathematical demonstration that of Navigation has always been my greatest favorite. With the strongest avidity and the strictest attention, I read almost all the different treatises written on the subject. I then made several voyages to sea, in which my land falls and other recurrences always agreed with my calculations; and here I had the pleasure of knowing that I not only understood the theory but also the practice of this art. Afterwards, I became a teacher of navigation and in the city of New York, I have followed that employment nearly these 10 years past. My method of demonstrating and illustrating the principles of navigation have answered my highest expectations; the nautical knowledge acquired by more than fifteen hundred of my pupils has given general satisfaction to their employers in almost every seaport in the United States. By what means this has been effected, may be seen in a small pamphlet written and published by myself entitled, *Exhibition of the Genuine Principles of Navigation* sold by T. and J. Swords, New York and W. P. Ferrand Philadelphia, price 25 cents. Although in teaching I have hitherto adopted the books in common use, yet more than 25 years ago I had and now have, in my own mind the firmest conviction that every writer on Common Navigation from the earliest to the present time, had and has utterly departed from the simplicity of nature, and involved the subject in a number of obscure intricacies which are altogether unnecessary and serve but to bewilder and perplex the learner.

In all our books on Navigation, it is easy to see the *Difference of Latitude*, made on any oblique course, is used only as an artificial means for finding the latitude of the ship. But I pledge myself to demonstrate that the ship's latitude may be more easily and more correctly found without having recourse to such a contrivance. Further, in *Middle Latitude*, and *Mercator's Sailing*, the *Departure, Middle Latitude, Meridional Difference of Latitude and Difference of Longitude*, are used as artificial mediums, either to find the ship's longitude or her course and distance to some known place. But here I again hold myself responsible to prove that both these two particular objects may be more easily and more correctly accomplished without the help of such artificial expedients. My intended publication will principally consist of a set of tables of about 400 pages quarto. The latitude and longitude sailed from, and the course and distance being given, these tables will show the ship's latitude and longitude merely by inspection and without the least calculation whatever. Again if the latitude and longitude of two places be given, my tables will by inspection, and independent of any calculation, show the course and distance from the one place to the other. All this is done to a far greater degree of accuracy than is attained by any of the methods now in use. The same tables will also assist land surveyors in finding the latitude and longitude of a place, when the course and distance to any other known place has been determined. The theory of my intended system is so simple, that it may be understood in a few minutes; and by means of the tables a person unacquainted with Navigation, may in five or six days be taught to keep a ship's reckoning at sea, in the most accurate manner.

New York, November 18th, 1806. G. BARON."

So far as we have been able to learn, nothing further is known of this egotistic, imperious, and sarcastic editor beyond what he has told us of himself on the covers and in the body of the *Mathematical*

Correspondent, the first born of American mathematical periodicals. Were there still extant copies of the first six numbers of the *Correspondent* containing covers, something more might be learned about him. However, he has told us enough of himself to convince us that he was mournfully lacking in those qualities of mind and heart so essential in carrying on successfully the enterprise he established.

Under the portrait of which we have spoken, it is stated that G. Baron was born September 1st, 1769.

With this number, as we have said, the *Mathematical Correspondent* under the Editorship of G. Baron and his associate editors ceased publication.

In 1807, appeared number 1, Vol. II, edited and published by Robert Adrain of Reading, Pa. The only copy of this number so far as is known is in the Library of the University of Pennsylvania. The following is the title page:

THE
MATHEMATICAL
CORRESPONDENT,
containing
NEW ELUCIDATIONS
Discoveries and Improvements,
In various branches of the
Mathematics,
With collections of
QUESTIONS
Proposed and Resolved
By Ingenious Correspondents.
——o——
VOL. II
utile dulci.

READING,
Printed for the Editor, Robert Adrain,
by Gottlob Jungmann.
1807
VOLUME II
Utile dulci

The preface is as follows:

The editor of this second volume of the *Mathematical Correspondent* has been induced to engage in the work from a thorough

knowledge of its utility in spreading and improving Mathematical Science; and he indulges himself in the hope that the real friends of science in the United States are sufficiently numerous and spirited to support him in the undertaking.

It is not necessary at present to enter into a lengthy defence of the practice of publicly proposing and answering new mathematical problems. Every one who has any acquaintance with the history of mathematics knows many valuable improvements and discoveries have resulted from the profound attention bestowed on the solution of new, curious, useful, or difficult problems. The greatest Mathematicians, as Pascal, Leibnitz, the Bernoullis, Huygens, Wallis, Newton, Maclaurin, Euler, Lagrange, Emerson, Simpson, Hutton, Vince, etc., have not disdained to enter the list, and try their strength of genius in contests of such a nature.

The truly ingenious method of differencing, *De Curva in Curvam** was invented by Leibnitz in attempting to resolve in a general manner difficult problems proposed by James Bernoulli. James Bernoulli himself invented a profound and general method of resolving isoperimetric problems in seeking the solution of the problems concerning the curve of swiftest descent, which was proposed by his brother John. According to Laplace and others a vast number of improvements sprang from the prize† problem proposed by the Academy of Sciences. And to close this enumeration, the immortal *Newton*, whose name I am unworthy to write or pronounce, was led to the discovery of the grand law of universal gravitation by his attempting to solve a new and curious problem proposed to him by Dr. Hooke.

But let us not sit down contented with imagining that those great men who have gone before us have exhausted the subject, and left us nothing to do, but to copy their writings. This were a dangerous as well as a groundless idea; and can never exist in the breast of a man of real genius. Doubtless many improvements and discoveries are still treasured up to reward the ingenuity of future enquirers: may it be our lot to obtain some portion of this precious deposit.‡

It is well known that various persons at different times may fall upon the same problems or solutions without any knowledge of their

*A problem or two exemplifying the method *de curva in curvam*, would probably be acceptable to many of our readers; they shall be gratified as soon as convenient.

†Perhaps the most elegant and profound discovery ever produced by a prize problem was that of Maclaurin in which he demonstrated for the first time that according to the known law of gravitation an oblate spheroid of uniform density would retain its figure of revolving uniformly about its less axis in a certain time which time he determined accurately in all cases.

‡We should however be exceedingly cautious in concluding that our researches are entirely new, merely because we have not met with those of a similar kind in the common authors on mathematics: many instances of precipitancy in this respect are well known to those who are acquainted with the progress of mathematical science.

mutual coincidence: it will not appear surprising therefore if questions which have been already discussed should sometimes make their appearance as new. The assistance of the contributors in this affair is requested by the editor: he hopes they will point out such questions already in print as coincide with those proposed from time to time as original problems. By this means we shall be enabled to learn more completely what real additions are made to the general stock of mathematical knowledge.

It will probably be expected from the present editor by some of his fellow contributors to the first volume, that he should make an apology for presuming to decide on the merits of their performances. Should such mathematicians as *Craig* and *Manghan** honor the *Mathematical Correspondent* with their contributions, he ingeniously confesses that nothing but their own consent would entitle him to the liberty of giving his judgment on their pieces.

Let it be considered however, that he, by no means, pretends to dictate to the Mathematicians of America.

Nec Sibi Regnandi Veniat Tam Dira Cupido

In deciding upon the comparative excellence of the pieces presented to him he wishes not to set up his own ideas as the standard of taste: he will merely give his judgment according to the extent of his skill; and leave to the public the task of finally determining the comparative ingenuity of rival competitors.

It would perhaps contribute something to the progress of science, if the editors were enabled by the sale of the work to have two Prize Questions in each number, a greater and a less. By this plan many who are not able to contend for a prize depending on certain abstruse researches might be usefully and honorably employed in resolving a problem of less profundity. On the other hand, Mathematicians of eminence, who would not accept a prize for what cost them scarcely a thought, might find in the problem of higher prize something worthy of attention.

It ought, however, to be indispensibly requisite in a prize question that it may be useful in improving some important theory little known, or in discovering some new one, or lastly in giving some rules of practical application. It is presumed our first prize question will be found useful in one of these ways; we hope, therefore, it will meet with the approbation and attention of judicious mathematicians.

*These two gentlemen were contributors to the first volume.—F.

(To be continued in March issue)

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Questions in Educating Mathematics Teachers for the Secondary School*

By EDWIN G. OLDS
Carnegie Institute of Technology

The education of teachers of high school mathematics is a matter of importance to all mathematicians. Many of the men and women who will extend the frontiers of mathematical knowledge during the latter half of the century are being taught in our schools today. The financial support necessary to permit them to engage in mathematical research will come largely from another group who must learn enough mathematics to become convinced of its value to civilization. We have a selfish interest in making sure both groups receive the best possible instruction.

It is the purpose of this paper to consider three questions which seem to be of importance in planning the education of a mathematics teacher: (1) How much mathematics should be required? (2) What kind of mathematics is desirable? (3) What "education" courses should be included? In attempting to supply a partial answer to each of these questions the author has been influenced by views expressed in articles listed in the bibliography at the close of this paper. (Bracketed numbers will be used to refer to these articles.)

When one considers how much mathematics should be required one finds that the average mathematician agrees that requirements for certification are too low. However, many of our high school teachers have majored in mathematics in collegiate institutions where the requirements far exceeded certification needs. It seems possible that a mistake is made in specifiying quantity rather than quality. If a student is interested enough in mathematics to choose the teaching

*A part of this paper was presented at a joint meeting of the Mathematical Association of America and National Council of Teachers of Mathematics, December 29, 1939, at Columbus, Ohio.

of it as a life work, he should evidence that interest by entering college with four years of high-grade high school mathematics. Then he could begin the study of analytics at once, omitting formal courses in college algebra and trigonometry, and making up the small shortage by individual study and by a later course in theory of equations.

This suggestion is not offered to provide time for another advanced course in mathematics, but to allow the election of some course which would add to the student's general culture. Adequate attention to culture is imperative because, as Mumford says, [13] p. 405, "As educators we must understand that the capitalist who knows only his markets, the engineer who knows only his machines, the teacher who knows only his books, are intellectually crippled people. The fatal weakness of their education and training is that it makes them incapable of dealing with the real world; they are helpless except in dealing with the series of abstractions in which they have achieved a minor competence." Of course, we do not intend the education of our prospective teachers to be lopsided, but after all of the desired sciences are crowded into the program there may be scant room for the humanities.

The question of "How much mathematics?" is linked with the question of "What type of mathematics is desirable?" It seems trite to repeat that the prospective teacher needs broad general training in zones which are the natural extension of the areas in which he expects to teach, and I find no disagreement with this general thesis. There may be some difficulty, however, especially in our smaller colleges, to find the body of courses which accomplishes this end. Administrative expedience may dictate the education of both teacher and specialist by means of the same set of courses; and then the curriculum probably is laid out for the best interests of the latter. It is high time that we accept the opinion of the Commission of 1935 (see [20] and [29]) that the mathematics curriculum designed to train the research specialist does not necessarily provide the best education for prospective teachers. We may imply that Dean Pittenger's statement, [8], p. 467, regarding college work in general, describes mathematics. He says, "The present departmentalized requirements may still fit the needs of prospective research specialists, and of students, especially in the sciences, who are looking forward to industrial employment. But they do not fit the needs of prospective teachers for positions below the senior college level." Statements by Wren [36], pp. 100-101, and Erskine [25], p. 34, would indicate the correctness of this implication.

In designing a curriculum for teachers we need to curb our desire to preserve the purity of mathematics. As Synge [26], p. 155, remarks,

"One of the facts which the historian of the future will not fail to note regarding our present epoch is the way in which mathematicians have turned from applied mathematics." We have considered our duty done with regard to applications because we have required courses in physics and the suggestion that any other field than physics might be closely related to mathematics and, therefore, tapped for applications, does not seem to meet with much approval. The fact that the physical applications with which the prospective mathematics teacher comes in contact mostly require a background beyond that possessed by the high school student of elementary mathematics, leaves the teacher without much in the way of enrichment material. It may be safe to assume that the average high school or college teacher knows little about the application of mathematics outside the field of physics. We subscribe to the belief that mathematics has many interesting uses but do not care to be embarrassed by a too severe cross-examination on specific illustrations. We are prone to criticize because certain professions and sciences do not require and use more mathematics; and we sometimes imply that they could use a lot more if they only knew it. It may be proper to suggest that mathematicians could aid in the more diverse and more efficient use of mathematics if they had more knowledge of the other professions and sciences. This latter statement is not meant as an unkind criticism of the general education of mathematicians but rather as a reason why it is so hard to find courses for prospective teachers which "get down to brass tacks" and teach the application of elementary high school mathematics in fields familiar to the high school student and in forms which will be significant to him. Most teachers are eager to learn of all possible applications of mathemtics but do not quite know how to do so. Those of us who have appealed to our colleagues in other departments know the difficulty in worming out of them such information in a form which we can understand.

When the high school teacher is engaged in teaching six or seven classes per day, to thirty or forty pupils per class, he has little time to search for applications outside of his textbook. Unless he is trained in application and has readily available sources, the probability is high that application will "go by the board". Articles like [16], [30], and [32] are a step in the right direction, but what the teacher of elementary mathematics needs is a compendium of applications, so that when he comes to such a subject as quadratic equations he can look up "Equations, quadratic" and find specific applications to problems of the home, of games, of civic affairs, of personal hygiene, etc. As far as the author knows, such a compendium is not in print, but, if some competent

mathematician would be willing to prepare such a reference work, he would make a significant and lasting contribution to mathematical education.

The recommendation by the Commission of 1935, [20], p. 276, that courses in theory of investment and mathematical statistics be required is a step in the right direction. The latter course, especially, has much of value which many mathematicians have failed to perceive because of confusing it with business statistics or "educational" statistics; courses which are judged to require little mathematics beyond arithmetic. While it is quite true that a course in mathematical statistics does provide some very much needed work in computation, that is but the beginning of the mathematical knowledge it uses and, therefore, strengthens. It demands from the student a thorough understanding of the principles of algebra, analytics, differential and integral calculus, and, for anything beyond the briefest course, the student will find that more than a bowing acquaintance with combinatory analysis, algebra of matrices, advanced calculus, point set theory, higher dimensional geometry and finite differences will be expected. The student will be shown applications, not only in economics and pedagogy, but also in such diverse fields as agriculture, ballistics, medicine, manufacture, and politics. Such a course cannot help but arouse his interest and broaden his knowledge in the whole gamut of sciences and near-sciences where experiments are performed, data tabulated, and the significance of hypotheses judged. Here is a field of mathematical application which, more than any other, provides fascinating problems for teachers at all stages of mathematical development, and gives some insight into the multifarious ways in which mathematics touches our lives from every direction.

In considering the third question, "What 'education' courses should be included?" it should be recognized that the average mathematician is not convinced that any are needed. However, it is quite possible that this attitude toward professional "education" courses is based on unfortunate experiences of ten or twenty years ago, and is not quite fair. Our present knowledge of the state of "education" may be as sketchy as the educationist's knowledge of mathematics. Perhaps the educationist has much to offer.

There is much merit in the opinion of the Commission of 1935 [20], p. 275: "We are inclined to think that, outside of foundation work in psychology, all of the theory of education presented to the candidate for a secondary teaching certificate in mathematics could best be given in courses definitely oriented with respect to his major teaching field and containing only students whose major or minor

interests are in this field. This method, though at variance with the common practice of presenting educational theory from a general viewpoint, is in successful operation at the University of Wisconsin in the training of teachers of secondary mathematics." The operation of this plan at the University of Wisconsin has produced very satisfactory results but, because of the cost, it has been temporarily discontinued.

The author believes that there should be a course on the teaching of secondary mathematics given by a mathematician who has knowledge of, and interest in, the principles of education. Secondly, an opportunity for practice in teaching mathematics under the supervision of a master teacher should be available. Furthermore there should be a brief course which will serve to orient the mathematics teacher with respect to his profession. Somewhere in this course should be found time to introduce the student to current literature on the teaching of mathematics, to acquaint him with the aims and publications of the several mathematical societies, and to instill desire to help, support, and promote the cause of mathematics, not only from a sense of duty, but also from one of interest. If he has not been taught them elsewhere, he needs some opportunity to learn the proper aims of public education, the techniques of testing and grading, the elements of guidance, how to get a job and keep it, and some homely facts about the relation between the public school and the tax payer.

In conclusion, the following list of courses is suggested as a possible answer to all three questions proposed in the introduction: analytic geometry, differential and integral calculus, theory of equations, history of mathematics, modern geometry, mathematical statistics, applications of elementary mathematics, teaching of secondary mathematics, practice teaching in mathematics, and professional orientation course. It is proposed that all of the above courses be given in the mathematics department, and as far as possible, be influenced by the future needs of the prospective teacher. Such a program would consume about one-third of the student's time.

To specify the outside requirements, we might mention English, public speaking, psychology, history, economics, physics, mechanics, and physiology, requiring at least one course in each field and another one-third of the student's time. The remaining third might be left free for the student to follow his own preference, hoping that he would be interested to include some further work in mathematics, but not requiring it.

It is the author's belief that such a program would produce teachers of high school mathematics who would be well-grounded in subject-

matter, skilled in technique, interested in mathematics and good salesmen of it, worthy of public respect, and possessed of adequate culture and balance for living happy private lives.

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Are There Any Questions?

By Peter Tullier
Fellow at Louisiana State University

It would be difficult to estimate the number of times this question is asked in the classrooms of colleges and universities in America. Again it would be difficult to estimate the number of questions brought on by this question.

But, speaking from my long experience as a student and my brief experience as a teacher, it seems that the number of responses to our original question definitely varies with the attitude of the questioner. In other words, some teachers have the ability to encourage students to ask questions; others lack this ability; still others have the positive ability to discourage students from asking questions.

It is very probable that all of us have studied under teachers in the three categories. For example, there is the one who, when asked a question by a student, turns to the youngster with a look of surprise and gives forth an answer to the effect that the student had better get his ninth grade algebra and review it. This type seldom encourages the student to ask further questions because of the ridicule involved.

Another "discouraging" type may be called the "bellower". This type answers the question, perhaps well, perhaps not so well; but the manner in which he answers does nothing towards encouraging other questions from the student.

Next might be mentioned the "wordy" type. This type of teacher will almost always answer the question; but in so doing, he talks so much that he gets off the track and never clears up the doubt in the student's mind.

On the other hand there is the teacher who, when asked a question, answers it with the "tact and diplomacy" of a statesman. This man considers the source of the question, always makes it seem important, and answers it with a clear, concise statement which removes the doubt from the mind of the student. This is the type of teacher whose students often ask questions; the type of teacher whose classes are most interesting; and consequently, the type of teacher whose students are most interested in their subject.

As a closing word, let me add that these few remarks were written under the assumption that the "blind" people in the class—those who "just can't see it"—(the word "it" meaning anything) and the "insincere" students—those who question for the sake of hearing themselves talk, are to be treated differently.

With the above assumption, how would you class yourself as an "answerer" to your "are there any questions?"

Mathematical World News

Edited by
L. J. Adams

The third annual William Lowell Putnam Mathematical Competition was scheduled to be held on March 2, 1940. The examination consists of two three-hour examinations. The questions are from calculus (with application to geometry and mechanics), higher algebra (determinants and theory of equations), elementary differential equations, and geometry, (advanced plane and solid geometry). Complete information concerning this annual event can be secured from Professor W. D. Cairns, Oberlin, Ohio.

The International Congress of Mathematicians, which was to have been held in Cambridge in September, 1940 has been postponed.

Professor Hubert E. Bray (Rice Institute) will present a course in *Fourier Series* at the summer session of University of California (Berkeley), from July 1, until August 9.

The American Mathematical Society will meet at Washington, D. C. on April 26-27, 1940. Two addresses (by invitation) have been scheduled. *Univalent and Multivalent Functions*. Professor Wladimir Seidel. *Complexes and Tensors*. Professor A. W. Tucker.

Professor W. D. Cairns (Oberlin College) has been given the title of professor emeritus.

The American Mathematical Society Colloquium Publications announces:

1. *Orthogonal Polynomials*. Professor Gabor Szego, Stanford University.
2. *Structure of Algebras*. Professor A. A. Albert, University of Chicago.

Professor G. D. Birkhoff (Harvard University) has been elected an honorary member of the Academia de Ciencias Exactas, Fisicas y Naturales de Lima.

The president of the Calcutta Mathematical Society for the year 1938 was Mr. B. M. Sen. The society met six times during that year, at which meetings sixteen papers were read.

The Council of the Mathematical Association of England has appointed an Executive Committee to transact the business of the Association, and has decided to cancel the annual meeting scheduled for January, 1940.

Oxford University (England) announces the publication of *The Mathematical Theory of Huygens Principle* by B. B. Baker and E. T. Copson.

Dr. Julian Coolidge (Harvard University) is giving a series of Lowell lectures in the public library in Boston. The subject of his lectures is *Topics in the History of Geometry.*

Dr. Eric Temple Bell (California Institute of Technology) addressed the Texas State Teachers Association in San Antonio during their recent annual convention. His subject was *What Mathematics Can Do for Human Beings.*

The National Council of Teachers of Mathematics was scheduled to meet in St. Louis on February 22, 1940. The program included the following addresses:

1. *Modern Youth Challenge the Curriculum.* J. Paul Leonard, Stanford University.
2. *Mathematics in Education as Viewed by a School Administrator.* John L. Bracken, Superintendent of Schools, Clayton, Mo.
3. *Mathematics in and for the Modern Curriculum.* W. D. Reeve, Teachers College, Columbia University.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and. naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 259. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle whose verbicenter lies on its incircle.

Solution by *C. W. Trigg*, Los Angeles City College.

The verbicenter is the intersection of the cevians to the points of contact of the escribed circles. Any verbisector, the incircle and its diameter perpendicular to the side cut by the verbisecter are concurrent. (No. 121, October, 1936, p. 56). Since each verbisector goes through the extremities of distinct diameters it follows that if the verbicenter is to lie on the incircle, then two verbisectors must meet the incircle again at the extremity of the third diameter. In the triangle ACB, let this point be V, the extremity of the indiameter $VD \perp BC$, and let AVE be the verbisector.

The distance from the vertex to the extremity of the diameter is equal to the distance from the verbicenter to the opposite side. (No. 172, February, 1938, p. 249), so

$$VE = AV = \tfrac{1}{2}AE = \tfrac{1}{2}\sqrt{\frac{s}{a}[a(s-a)+(b-c)^2]}\ .$$

Now $\quad ED = b-c$ and $VD = 2r = 2\sqrt{(s-a)(s-b)(s-c)/s.}$

$\overline{VD}^2 + \overline{ED}^2 = \overline{VE}^2$. When the indicated values are substituted in this equation and the result is simplified, we have

$$(3a-b-c)\{[a-(b+c)]\,[5a^2+a(b+c)+(b-c)^2] \\ -(11a+b+c)(b-c)^2\} = 0.$$

The sum of two sides of a triangle is always greater than the third side, so the second factor is always negative, hence $3a=b+c$, which condition must be met in order that the verbicenter fall on the incircle. To construct such a triangle on some arbitrary base a, with a radius b such that $a<b<2a$ and one extremity of a as center describe a circle. With the other extremity as center and a radius equal to $3a-b$ describe another circle. The intersections of the circles determine the alternate third vertices of the required triangle. It will be noted that the locus of A for a fixed a is an ellipse with semi-axes $3a/2$ and $a\sqrt{2}$.

This same result may be secured by considering the verbisector BVM which passes through F, the end of the indiameter $FIN \perp AC$. Draw $IP \perp VF$ so $FV=2FP$. Triangles FPI and FNM are similar so $FI:FP::FM:FN$. Hence $FP=FI\cdot FN/FM=2r^2/FM$. Now $BF/BM=(s-b)/s$, so $FM/BM=b/s$ (No. 172), hence

$$FV=4r^2/FM=4r^2s/b\cdot BM.$$

But $FV=BM-2BF=BM-2BM(s-b)/s=BM(2b-s)/s$. When these two values of FV are equated, the value of

$$BM=\sqrt{\frac{s}{b}[b(s-b)+(c-a)^2]}$$

substituted, and the result is simplified, we have $(s-b)^2=(c-a)^2$ which reduces to $3a=b+c$ as in the first method.

No. 304. Proposed by *Yudell Luke*, University of Illinois.

A and B are two parabolas $y^2=2px$ and $x^2=2qy$ respectively, whose points of intersection are the origin and C. The tangents from C to A and B meet B and A at the points D and E respectively. Show that DE is a common tangent of A and B.

Solution by *W. L. Roberts*, Student, Colgate University.

It is seen that the two parabolas are similar and symmetric in notation. A simultaneous interchange of x and y, and p and q, turns one equation into the other.

The coordinates of the point C are found to be $(2p^{1/3}q^{2/3}, 2q^{1/3}p^{2/3})$, and the equation of the tangent CE is $2p^{1/3}q^{2/3}x=qy+2q^{4/3}p^{2/3}$. Then the coordinates of the point E are $(\frac{1}{2}q^{2/3}p^{1/3}, -q^{1/3}p^{2/3})$.

Whence the equation of the line tangent to the parabola A at E is $q^{1/3}p^{2/3}x+q^{2/3}p^{1/3}y+\frac{1}{2}qp=0$.

It is now seen that by interchanging at the same time x and y, and p and q, this tangent equation is unchanged. Therefore the line is the common tangent to the two parabolas at the symmetric points E and D.

Also solved by *Karleton W. Crain, Henry G. Diebel, Albert Farnell, D. L. MacKay, Johannes Mahrenholz, Charles Templeton, P. D. Thomas, C. W. Trigg,* and the *Proposer.*

No. 305. Proposed by *H. S. Grant,* Rutgers University.

There are $n+3$ urns, $U_1, U_2, \cdots, U_{n+3}$. Each of the urns $U_1, U_2, \cdots, U_{n+1}$ contains a white and b black balls, $n \leqq a+b$; each of the urns U_{n+2}, U_{n+3} contains c white and d black balls. n balls are drawn, one at a time, from U_1 and placed in U_{n+2}; the $c+d+n$ balls in U_{n+2} are then thoroughly mixed. One ball is drawn from each of the n urns $U_2, U_3, \cdots, U_{n+1}$ and placed in U_{n+3}; the $c+d+n$ balls in U_{n+3} are then thoroughly mixed.

Show that the probabilities that a ball extracted from U_{n+2} be white or black are the same as the corresponding probabilities for a ball extracted from U_{n+3}, and find simple expressions for these probabilities.

Solution by the *Proposer.*

The probabilities that a ball extracted from U_{n+2} be white or black are

$$P_1 = \frac{1}{(a+b)_n(c+d+n)} \sum_{i=0}^{n} \binom{n}{i} a_{n-i} b_i (c+n-i)$$

and
$$Q_1 = \frac{1}{(a+b)_n(c+d+n)} \sum_{i=0}^{n} \binom{n}{i} a_{n-i} b_i (d+i),$$

respectively, where $y_r = y(y-1)(y-2)\cdots(y-r+1)$ for $r>0$, $=1$ for $r=0$;

$$\binom{s}{t} = s!/\{t!(s-t)!\}.$$

The corresponding probabilities for a ball extracted from U_{n+3} are

$$P_2 = \frac{1}{(a+b)^n(c+d+n)} \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i (c+n-i)$$

and
$$Q_2 = \frac{1}{(a+b)^n(c+d+n)} \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i (d+i),$$

respectively. Since P_1+Q_1 and P_2+Q_2 must each be unity, it is only necessary to show that $P_1=P_2$. We shall however establish directly that $P_1=P_2$ and $Q_1=Q_2$, and use $P_1+P_2=Q_1+Q_2=1$ as a check.

Using $(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$ and $(a+b)_n=\sum_{i=0}^{n}\binom{n}{i}a_{n-i}b_i$,* we have

$$P_1=\frac{1}{c+d+n}\left\{c+\frac{na}{(a+b)_n}\sum_{i=0}^{n-1}\binom{n-1}{i}(a-1)_{n-1-i}b_i\right\}$$

$$=\frac{1}{c+d+n}\left\{c+\frac{na(a+b-1)_{n-1}}{(a+b)_n}\right\}=\frac{c(a+b)+na}{(a+b)(c+d+n)}.$$

$$P_2=\frac{1}{c+d+n}\left\{c+\frac{na}{(a+b)^n}\sum_{i=0}^{n-1}\binom{n-1}{i}a^{n-1-i}b^i\right\}=\frac{c(a+b)+na}{(a+b)(c+d+n)}$$

$$Q_1=\frac{1}{c+d+n}\left\{d+\frac{nb}{(a+b)_n}\sum_{i=0}^{n-1}\binom{n-1}{i}a_{n-1-i}(b-1)_i\right\}$$

$$=\frac{1}{c+d+n}\left\{d+\frac{nb(a+b-1)_{n-1}}{(a+b)_n}\right\}=\frac{d(a+b)+nb}{(a+b)(c+d+n)}.$$

$$Q_2=\frac{1}{c+d+n}\left\{d+\frac{nb}{(a+b)^n}\sum_{i=0}^{n-1}\binom{n-1}{i}a^{n-1-i}b^i\right\}=\frac{d(a+b)+nb}{(a+b)(c+d+n)}$$

$$P_1+Q_1=P_2+Q_2=\frac{c(a+b)+na+d(a+b)+nb}{(a+b)(c+d+n)}=1.$$

No. 306. Proposed by *C. N. Mills*, Illinois State Normal University.

Without locating the points of contact, find the equation of the conic touching the lines $x=0$, $y=0$, $x=2$, $x-y+1=0$, $x+y-3=0$.

Solution by *C. W. Trigg*, Los Angeles City College.

It is evident upon inspection that the last two lines intersect at (1, 2) and that the five lines which determine the conic form a pentagon symmetrical to $x=1$. Hence the required conic is the inscribed ellipse, and is of the form

$$\left(\frac{x-1}{1}\right)^2+\left(\frac{y-b}{b}\right)^2=1.$$

*Vandermonde's Theorem.

In this equation put $x=y-1$ and secure

$$(b^2+1)y^2-(4b^2+2b)y+4b^2=0.$$

The discriminant of this quadratic is equal to zero, since $x-y+1=0$ is tangent to the conic, so $b^2(3-4b)=0$, whence $b=3/4$. The required equation is therefore

$$9(x-1)^2+(4y-3)^2=9, \text{ or}$$

$$9x^2+16y^2-18x-24y+9=0.$$

Also solved by the *Proposer*.

No. 307. Proposed by *E. P. Starke*, Rutgers University.

It is known that there are infinitely many pairs of consecutive squares whose sum is a square. What is the smallest value of $k>2$, such that the sum of the squares of k consecutive positive numbers can be a square?

Solution by *J. Rosenbaum*, Bloomfield, Connecticut.

The sum of the squares of the first n integers decreased by that of the first m integers is

$$n(n+1)(2n+1)/6-m(m+1)(2m+1)/6.$$

With k written for $n-m$, the condition that the sum of the squares of k consecutive positive integers, starting with $m+1$, be a square is accordingly

$$(1) \qquad k(2k^2+6km+3k+6m^2+6m+1)=6x^2.$$

Now the smallest value of $k>2$ for which (1) has solutions is 11. For $k<11$, the details follow.

For $k=3$: (1) becomes $3m^2+12m+14=x^2$, which requires that x^2 be of the form $3a+2$, an impossibility. For $k=4$: (1) shows that x is even, hence the right member contains the factor 8; but this is impossible since the number in parenthesis is odd. For $k=5$: (1) becomes $5[(m+3)^2+2]=x^2$, which is impossible since $(m+3)^2+2$ is never a multiple of 5. For $k=6$: (1) becomes $6m^2+42m+91=x^2$. Whether m is odd or even the left member is of the form $4a+3$, which is impossible for a square.

For $k=7$: (1) becomes $7(m^2+8m+20)=x^2$. For odd m, the left member is of the form $4a+3$ which is never a square. For $m=2m'$ and $x=2x'$, the equation reduces to $7(m'^2+4m'+5)=x'^2$. The left member is now of the form $4a+3$ or $4a+2$ according as m' is even or odd; but neither of these is possible for a square.

For $k=8$: (1) may be reduced to $2m^2+18m+51=(x/2)^2$. Whether m is even or odd, this left member is of the form $4a+3$ and hence is not a square. For $k=9$: (1) implies that x is a multiple of 3; thus the right member is divisible by 27 while the left is not. For $k=10$: (1) becomes $5(2m^2+22m+77)=x^2$, in which it is seen that the number in parenthesis must be a multiple of 5; but for no value of m is this possible.

When, however, $k=11$, the substitution $x=11u$, $m=v-6$ reduces (1) to

$$v^2+10=11u^2 \tag{2}$$

which is satisfied by $v=1$, $u=1$. Furthermore if

$$v'=|10v\pm 33u|, \qquad u'=|3v\pm 10u|, \tag{3}$$

$(u'\ v')$ will be a solution of (2) whenever (u, v) is. Thus there are infinitely many solutions* and hence infinitely many sets of integers satisfying the proposed problem. The simplest examples are

$$18^2+19^2+\cdots+28^2=77^2,$$

$$38^2+39^2+\cdots+48^2=143^2.$$

Also solved by *C. W. Trigg.*

No. 308. Proposed by *Albert Farnell,* Louisiana State University.

Sum the infinite series whose representative term is $n^2/(n-1)!$

Solution by *Gaines B. Lang,* Emory University, Georgia.

Let us define functions

$$f_k(x) \ = \ \sum_{n=1}^{\infty} \frac{n^k x^n}{(n-1)\ !}, \ k=0, 1, 2, 3, \cdots \ .$$

Then the series defining $f_k(x)$ converges uniformly in any finite region. In this notation, the proposed problem is simply to find $f_2(1)$. We have at once $f_0(x)=xe^x$, and by term by term differentiation we obtain $xf'_k(x)=f_{k+1}(x)$. Hence we have

$$f_1(x)=xf'_0(x)=e^x(x^2+x),$$

$$f_2(x)=xf'_1(x)=e^x(x^3+3x^2+x).$$

*It is not difficult to show that all solutions of (2) are given by repeated use of (3) starting with $u=1$, $v=1$. Discussion similar to the above shows that the numbers $k<100$ having the analogous property are 2, 11, 23, 24, 26, 33, 47, 49, 50, 59, 73, 74, 88, 96. See also Problem No. 3917 in the American Mathematical Monthly, May, 1939, the solution of which is not yet published.—ED.

Thus $f_2(1)=5e$.

Also solved by *H. M. Gehman*, *J. Mahrenholz*, *C. W. Trigg*, and the *Proposer*.

No. 311. Proposed by *Leo F. Epstein*, Massachusetts Institute of Technology.

Prove the relation

$$\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{t=1}^{\infty}\frac{(\ln t)^m}{t!}=1.$$

Solution by the *Proposer*.

Both the sums being absolutely convergent, the order of summation may be reversed. Thus:

$$\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{t=1}^{\infty}\frac{(\ln t)^m}{t!}=\sum_{t=1}^{\infty}\frac{1}{t!}\sum_{m=1}^{\infty}\frac{(\ln t)^m}{m!}$$

$$=\sum_{t=1}^{\infty}\frac{1}{t!}(e^{\ln t})=\sum_{t=1}^{\infty}\left(\frac{t}{t!}-\frac{1}{t!}\right)=e-(e-1)=1.$$

Also solved by *J. C. Stewart*, Louisiana State University.

No. 309. Proposed by *W. V. Parker*, Louisiana State University.

If three lines $a_ix+b_iy+c_i=0$ are such that

$$a_1+a_2+a_3=b_1+b_2+b_3=0,$$

show that the area of the triangle formed by them is $|(c_1+c_2+c_3)^2/2\Delta|$, where $\Delta=|a_ib_i|$.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

From $a_1+a_2+a_3=b_1+b_2+b_3=0$, we have

$$\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=-\begin{vmatrix}a_2+a_3 & b_2+b_3\\ a_2 & b_2\end{vmatrix}=\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix}$$

$$=\begin{vmatrix}a_2+a_3 & b_2+b_3\\ a_3 & b_3\end{vmatrix}=-\begin{vmatrix}a_1 & b_1\\ a_3 & b_3\end{vmatrix}=\begin{vmatrix}a_3 & b_3\\ a_1 & b_1\end{vmatrix}.$$

Furthermore,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+a_2+a_3 & b_1+b_2+b_3 & c_1+c_2+c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= (c_1+c_2+c_3) \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The area of the triangle is*

*See the Note attached to the solution of No. 260, this Magazine, April, 1939, p. 345.—ED.

$$\frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2}{2 \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}}$$

$$= (c_1+c_2+c_3)^2/2 \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Also solved by *Johannes Mahrenholz*, *C. W. Trigg*, and the *Proposer*.

No. 310. Proposed by *H. A. Simmons*, Northwestern University.

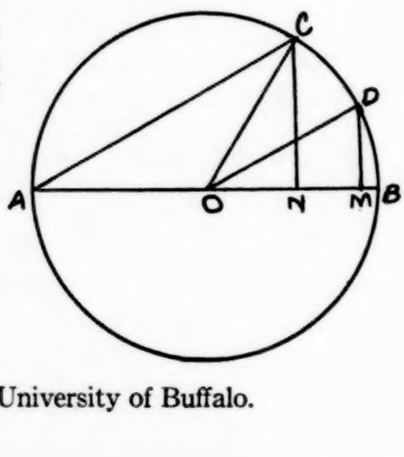
Derive the formula

$$\tan(x/2) = \sin x/(1+\cos x)$$

without the use of any irrational expression.

Solution by *Janet Rung*, D'Youville College, Buffalo, New York.

Let AB be the diameter of the unit circle, angle $BOA = x$, angle $BOD = x/2$. Draw CA and drop CN perpendicular to the diameter.



Then angle $A = x/2$,

$$CN = \sin x, \quad ON = \cos x.$$

Accordingly,

$$\tan(x/2) = \sin x/(1+\cos x).$$

Solution by *Harry M. Gehman*, University of Buffalo.

$$\sin x/(1+\cos x)=\frac{2\sin(x/2)\cos(x/2)}{1+2\cos^2(x/2)-1}=\frac{\sin(x/2)}{\cos(x/2)}=\tan(x/2).$$

Solution by the *Proposer*.

$\sin(x/2)=\sin(x-x/2)=\sin x\cos(x/2)-\sin(x/2)\cos x$. Thus

$$(1+\cos x)\sin(x/2)=\sin x\cos(x/2)$$

and

$$\tan(x/2)=\sin x/(1+\cos x).$$

Also solved by *Annie M. H. Christensen, Albert Farnell, R. E. Greenwood, Jr., Mark Heyman,* and *C. W. Trigg.*

In connection, see method employed by *Roscoe Woods*, Am. Math. Monthly, p. 174, March, 1936.

PROPOSALS

No. 339. Proposed by *Virgil S. Mallory*, New Jersey State Teachers College at Montclair.

Jack Donovan was killed on a lonely road two miles from Trenton, New Jersey, at 3:30 A. M., March 17, 1933. Shorty Malone, Tony Verelli, Hank Rogers, Joey Freiberg and Red Johnson were arrested a week later in Philadelphia, Penn. Under cross-examination, each of these men made four simple statements, three of which were true, and one false. One of these men killed Donovan: deduce which.

Shorty said: I was in Chicago when Donovan was murdered. I never killed anyone. Red is the guilty man. Joey and I are pals.

Hank said: I did not kill Donovan. I never owned a revolver in my life. Red knows me. I was in Philadelphia March 17th.

Tony said: Hank lied when he said he never owned a revolver. The murder was committed on St. Patrick's day. Shorty was in Chicago at that time. One of us five is guilty.

Joey said: I did not kill Donovan. Red has never been in Trenton. I never saw Shorty before. Hank was with me in Philadelphia March 17th.

Red said: I did not kill Donovan. I have never been in Trenton. I do not know Hank. Shorty lied when he said I'm guilty.

No. 340. Proposed by *V. Thébault*, Le Mans, France.

Find such a four-digit number that when 385604 is written at its right the result is a perfect square.

No. 341. Proposed by *E. P. Starke*, Rutgers University.

The intersections of two parabolas whose axes are perpendicular are concyclic.

No. 342. Proposed by *Althéod Tremblay*, Québec, Canada.

A well known problem in algebra is as follows: "An American city has m streets running east and west and n avenues running north and south. In how many ways can a man go from the southwest corner to the northeast corner, if he always faces either north or east?" The answer is $(m+n)!/(m!\cdot n!)$.

This problem suggests the following question. If the man of the problem chooses his path at random, what is the probability of his meeting a friend who is waiting at the corner of Avenue P and StreetQ?

No. 343. Proposed by *Robert C. Yates*, Louisiana State University.

Given in the plane a point P and a line segment a with P not upon a. Outlawing the use of the compasses as a pair of dividers, construct with straightedge and compasses the circle with center at P and radius a.

No. 344. Proposed by *C. Bull Foster*, Evanston, Illinois.

Prove: If $\{a_n\}$ is a sequence of complex terms such that

$$\frac{a_n}{a_{n+1}}=1+\mu/n+O(1/n^\lambda), \quad \lambda>1, \quad \mu=a+ib,$$

where a, b, λ are real and $a>0$, then

$$\sum (a_n - {}_pC_1a_{n+1} + {}_pC_2a_{n+2} - \cdots \pm a_{n+p})$$

is absolutely convergent for $p=1, 2, 3, \cdots$.

Bibliography and Reviews

Edited by
H. A. SIMMONS

New First Course in the Theory of Equations. By Leonard E. Dickson. John Wiley & Sons, Inc., New York, 1939. ix + 185 pages. Price $1.75.

New First Course—excels *First Course*—principally on account of improvement in worked examples and addition of an abundance of well chosen problems proposed for the student. However, the *new* text contains also improved treatments of *constructions with ruler-and-compasses, Sturm's Theorem,* and *solutions of systems of linear equations.* For the latter subject, the *new course* gives the key-theorem, namely: *a necessary and sufficient condition for the consistency of a system of linear equations is that its coefficient matrix and augmented matrix have the same rank.*

Having found *First Course*... a good textbook and aware of the niceties that are added in *New First Course*..., the writer is convinced that the latter is an extremely good textbook.

Very minor details (of relatively small significance pedagogically) regarding which the writer would suggest changes are fairly frequent in the *New First Course*...; some of these are as follows. Page 7, line 8, replace *the* before equation by *an;* page 13, Example 3, rewrite a part of the line

$$\frac{f(x)}{x^2+x-2} \equiv x^2+x-6=0, \quad \text{roots 2 and } -3;$$

page 28, line 25, replace *a* before *integer* by *an;* page 50, line 8, insert *is* or *may be* before *shown;* page 76, line 5, replace *would* by *should;* page 129, line 1, omit the first word, *identically,* which immediately follows the symbol $\equiv$; page 154, line 11, modify or omit the sentence, "The following example and problems do not require or illustrate any of our preceding or later results and hence may be omitted". Some of the problems in question afford good uses of *elimination,* which is explained just before the quoted sentence.

These minor details and many others in the book may be regarded as a sort of *stream-lining;* they were no doubt such in the minds of

Professor Dickson and several "experts", who read the manuscript of this text.

The format of the book is excellent, and it contains very few misprints.

Northwestern University. H. A. SIMMONS.

Proofs and Solutions of Exercises Used In Plane Geometry Tests. By Hale Pickett, Bureau of Publications, Teachers College, Columbia University Contributions to Education, No. 747. $1.60.

Within recent years considerable attention has been given to the selection of what might be called the "essential" theorems of plane geometry. Dr. Pickett has studied this problem with great thoroughness and his "revised list of theorems" is derived from a most careful analysis of the geometry examinations given by certain public and state examining bodies during the period from 1923 to 1935, inclusive. He solved the 3002 questions given in these examinations and analyzed the solutions to discover the basis theorems, constructions, postulates, axioms, and definitions as well as the algebraic techniques and methods of proof used. The frequency of these theorems is then studied and a percentile rating is developed. Of the 125 theorems, for example, which the author considers "basic" 71 fall between the 0-10 percentiles while only 4 fall between the 80-100 percentiles. Accepting the utility of these theorems in solving the problems of the given examinations as one criterion of reliability, the author accepts the proposition that "theorems known to have a greater utility should receive greater emphasis." For the purpose of establishing his "revised list of theorems" he adds to this criterion of utility two other criteria which are (1) Relative importance as indicated by lists of theorems in recognized syllabi, and (2) Persistence in representative texts. On applying these criteria, he recommends a "revised list of theorems" consisting of 58 propositions. These theorems together with 18 constructions and the necessary postulates, axioms, and definitions are considered as basic to the study of demonstrative geometry.

Ohio State University. HAROLD FAWCETT.

Synthetic Projective Geometry. By R. G. Sanger, New York, McGraw-Hill Book Company, Inc., 1939. 9+175 pages.

As is stated in the preface, the arrangement of material in this book roughly follows that of J. W. Young's monograph on projective geometry. The expansion of the content of this monograph by the

inclusion of exercises and in some cases additional material and more complete proofs is, in the reviewer's opinion, a very worthy contribution to mathematical literature. Although some teachers may wish that there were more exercises, on the whole the choice made by Sanger in covering additional materials is more satisfactory at least from the point of view of those who are preparing teachers of secondary mathematics.

The text contains sufficient material for a short course in projective geometry to cover three semester hours or four quarter hours. For such a course, which requires about all of the time that the average undergraduate or even first year graduate student can give to this subject, Sanger's text has a great many advantages. The sense of completeness which comes from covering the whole of the material is in itself a desirable factor.

Particularly for those preparing to teach in the secondary schools the organization of materials is ideal. An acknowledgment of assumptions and undefined terms is to be commended. Also the frequent references to several means of developing certain topics, although only one is presented, with statements such as "æsthetically, this would be a desirable thing to do" or "logically, this mode of attack is not so desirable as the former but, at least for a first course, is sometimes more practical", are of value to the prospective teacher of geometry. If, as some maintain, *synthetic projective geometry* can be made more valuable for the teachers of plane geometry than the so-called *college geometry*, this text is a definite step forward.

The brief section on *synthetic space geometry*, which incidentally is a valuable addition not included in the Carus monograph, and the historical sketch of the last chapter should bring praise to the author. The reviewer finds the arrangement under which most of the metric properties are introduced in Chapter VII very satisfactory. No attempt is made to introduce analytic methods on a purely projective basis as in the eighth chapter of the monograph.

The omission of all dual proofs and the consequent avoidance of the "double column" presentation is a step in the right direction. The fact that the duals are left as exercises for the students makes the inclusion of more exercises less desirable. There is scarcely a better aid to learning for the student than that afforded by his own formulation of dual proofs. Full advantage should be taken of the satisfaction that comes to the student when he appreciates his own power to prove theorems and constructions.

It is because of the faith of this writer in the value of a course in *synthetic projective geometry* for the undergraduate mathematics major

that he is so pleased with the text. No other undergraduate course can give so easily an understanding of the nature of mathematics as the course in projective geometry; and because it depends less on skills which some of the majors often lack, not infrequently projective geometry provides a basis for a new or renewed inspiration for the prospective mathematics teacher which will soon be of vital importance to him in his teaching.

Southern Illinois Normal University. JOHN R. MAYOR.

First Course in Theory of Numbers. By H. N. Wright. John Wiley & Sons, New York, 1939, vii+108 pages. Price $2.00.

No doubt a great many teachers of elementary number theory have felt the need for a short text, easy enough for undergraduate students to read, and containing at least some exercises which such students can solve without aid. Professor Wright has written such a book.

Although this text is intended for a one-semester course, it is not likely that the average class could finish the text in that time without omitting parts of it. It is arranged so that suitable omissions could be made.

The book contains five chapters and a table of prime numbers from 2 up to 2741.

Chapter I treats divisibility, including Euclid's algorithm and its applications to greatest common divisor and to the solution of the Diophantine equation $ax+by=c$. The author gives a neat derivation of the formula for $\Phi(m)$ by use of mathematical induction, and then discusses the solutions of the equation $\Phi(x)=n$. Chapter I also contains sections on the unique factorization theorem, the infinitude of primes, and perfect numbers.

Chapter II covers expansion of rational and irrational numbers into simple continued fractions, with proofs of some approximation theorems and some theorems about periodic continued fractions. There is also an application of this theory to the solution of the equation $x^2-Dy^2=N$. The author gives in this chapter a very convenient way of arranging the numerical work in finding successive convergents.

Chapter III contains the usual fundamental theorems on congruences, together with Fermat's theorem, Wilson's theorem, congruences of higher degree, simultaneous congruences, power residues, primitive roots, and indices.

Chapter IV is devoted to quadratic residues, including the Jacobi symbol, the lemma of Gauss, the reciprocity law, and the general

quadratic congruence. It also contains an application of quadratic residues to the factorization of large numbers; a numerical example is included.

The last chapter consists of an extremely brief discussion of Diophantine equations which includes only a few standard types. There is one section on Fermat's last theorem.

The author makes no attempt to treat quadratic forms, since they would require more time than could be given in one semester.

There are a few mis-prints, including one on page 21 in which several equations are written with a capital C instead of a small c. Also, on page 16, in the center of the page, the period between $1/a_n$ and a_i could easily be mistaken for a multiplication sign, since there is no etra space between the symbols.

Although the reviewer noticed no objections to proofs except such as could be removed by a few simple remarks, he did feel that some mention of mathematical induction would have been desirable in theorem 2 on page 4, in corollary 2 on page 5, and in the corollary to theorem 6 on page 45. On page 90, just below equation (4′), the author makes a statement which is true except in the trivial case in which two corresponding coefficients in the two equations are both 0. The example $2x+3y+0z-5=0$, $x+2y+0z-3=0$ is an illustration of this trivial case.

The reviewer does not intend to give the impression that the book has many faults. On the contrary, he feels that it is quite suitable for the type of course for which it was intended.

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General Analysis, Part II. The Fundamental Notions of General Analysis. Eliakim Hastings Moore, with the cooperation of Raymond Walter Barnard, 1939. The American Philosophical Society, Philadelphia, 1939. vi+255 pages.

The analytic foundations of the second General Analysis theory of E. H. Moore are presented in this volume. The algebraic foundations of the theory were presented in Part I. On pages 7-12 and 16-18 of the Introduction in Part I, there is an adequate summary, in remarkably untechnical language, of the contents of these first two of the four parts in which it is planned to publish the theory. These volumes merit a place in mathematical libraries, both private and institutional.

The highly notational character of Moore's work has been retained, but the results are also continually summarized in precise English. Indeed, each chapter has an introductory exposition, in

non-notational language, of the contents of that chapter. These chapter-introductions can be read independently of the remainder of the text.

There is very great charm and inspiration in the generality and final simplicity of Moore's research. The importance of the generality of the general range P was stressed in Part I. In Part I also is the general reciprocal of a matrix which is not necessarily square and which is not necessarily non-singular. In Part II the very general number systems of Part I are further restricted by postulates to a number system which possesses continuity properties suitable for the limit process to be introduced. This number system is proved to be simply isomorphic with either the real numbers, the complex numbers, or real quaternions. Another instance of generality and simplicity in Part II is Moore's general limit. It particularizes to the ordinary sequential limit, to the limit of a function of one variable, to ordinary multiple limits, and to the limit as to a norm common in theories of integration. Other instances of generality are the modular space determined by a positive Hermitian matrix ε, the integral operator J determined by ε, and the modular matrices determined by ε. There are various modes of convergence of a set of modular vectors to a limit, introduced to insure that the modular space should be closed as to the limit process and that the modulus of the limit should be the limit of the moduli. Also various modes of convergence are used for sets of modular matrices. There is continuity and complete continuity of functional operators and of linear transformations. A final instance of generality and simplicity is the relations between linear continuous functional operators and transformations on the one hand and the J operator on the other.

The high standard of excellence of organization and presentation set in Part I has been maintained by Professor Barnard. To him the mathematical world is deeply indebted.

Northwestern University. L. W. GRIFFITHS.